ROMANIAN JOURNAL OF INFORMATION SCIENCE AND TECHNOLOGY
Volume **10**, Number 2, 2007, 123–136

Mathematical (Denotational) Semantics of Some Reducts of Ambient Calculus and Brane Calculi

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Abstract. Mathematical models of some reducts of Ambient Calculus [4] and Brane Calculi [2], [3] are presented and discussed. The models are certain classes of those hereditary finite sets which represent some membrane systems; the notion of a hereditary finite set was introduced in [1] and the construct of a membrane system is a basic concept of membrane computing [10].

1. Introduction

We present and discuss certain mathematical models of some reducts of Ambient Calculus [4] and Brane Calculi [3] aiming to describe the interconnections between these calculi and membrane computing [10] on semantical (model theoretical) level with mathematical precision.

The models are determined by valuations of expressions of the languages of considered calculi in certain classes of hereditary finite sets. The notion of a hereditary finite set introduced in [1] was already applied in general computation theory, cf. [6], and in membrane computing, cf. [7], [8].

The valuations are functions inductively defined with respect to the construction of expressions in a similar way as terms and formulas of predicate calculi are valuated in relational systems such that the valuations assign to expressions the elements of model domains (universes); these elements (assigned to expressions) are meant as denotations of expressions or constructs denoted by expressions, respectively, via valuations.

The discussed models besides providing a denotation of expressions of the languages of considered calculi via valuations have also some practical interpretation.

Namely, the elements of model domains represent certain membrane systems (thus the expressions of considered calculi denote some membrane systems via valuations) which can be used for the investigations of model theoretical aspects of Ambient Calculus by using membrane computing methods and to describe the links between membrane computing and Brane Calculi, already pointed out and discussed in [5], in some mathematical way on semantical (model theoretical) level.

In Section 2 of the paper we recall the notion of a hereditary finite multiset introduced in [7] as a special case of hereditary finite sets and we introduce a notion of a sequential hereditary finite multiset (an extension of the notion of hereditary finite multiset). Then in Section 2 we discuss algebras (commutative monoids) of hereditary finite multisets and sequential hereditary finite multisets to use them for introducing models of considered calculi.

In Section 3 we introduce models of (some reducts of) considered calculi by defining valuations of expressions of the languages of considered calculi in the monoids of hereditary finite multisets and of sequential hereditary finite multisets discussed in Section 2. Then we show a completeness of the valuations by means that if the valuations of two expressions are equal, then these expressions are structurally congruent for an appropriate structural congruence defined like in [2], [3], [4]. We give also hints how to model systems or processes containing unbounded number of parallel replicas of some their parts discussed in [2], [3], and [4].

In Appendix we recall the notion of a weak membrane system (some simple generalization of membrane systems from [10]) and some part of representation of weak membrane systems by hereditary finite multisets presented in [7].

2. Hereditary finite multisets, sequential hereditary finite multisets, and their algebras

In this section we recall the notion of a hereditary finite multiset introduced in [7] and then we define a new notion of a sequential hereditary finite multiset. We introduce also algebraic operations of sum and subtraction of hereditary finite multisets and sequential hereditary finite multisets.

The notion of a hereditary finite multiset is a special case of the notion of a hereditary finite set introduced in [1].

We recall the notion of a hereditary finite set.

For a potentially infinite set L of labels or names which are *urelements*, i.e., they are not (treated as) sets themselves, we define inductively a family of sets HF_i for natural numbers $i \geq 0$ such that

$$\mathrm{HF}_0 = \emptyset,$$
$$\mathrm{HF}_{i+1} = \text{the set of nonempty finite subsets of } L \cup \mathrm{HF}_i.$$

The elements of the union $\mathrm{HF} = \bigcup\{\mathrm{HF}_i \mid i \geq 0\} \cup \{\emptyset\}$ are called *hereditary finite sets over* L or *hereditary finite sets with urelements* in L, or simply *hereditary finite sets* if there is no risk of confusion.

For $x \in \mathrm{HF}$ we define its weak transitive closure $\mathrm{WTC}(x)$ by

$$\mathrm{WTC}(x) = \bigcup \big\{ \mathrm{WTC}(y) \mid y \in x \text{ and } y \in \mathrm{HF} \big\} \cup \{x\}.$$

We recall now the notion of a hereditary finite multiset introduced in [7].

For a set $\mathcal{O}$ of objects and the set N^+ of natural numbers excluding 0 we consider urelements belonging to a coproduct (disjoint sum) $N^+ \dot\cup (N^+ \times \mathcal{O})$, where we adopt the convention that for a natural number $n \in N^+$ and for an ordered pair $(m, a) \in N^+ \times \mathcal{O}$ the urelements in $N^+ \dot\cup (N^+ \times \mathcal{O})$ corresponding to n and (m, a), respectively, are denoted by $\langle n \rangle$ and by $\langle m, a \rangle$.

Let x be a hereditary finite set with urelements in $N^+ \dot\cup (N^+ \times \mathcal{O})$. We say that x is a *hereditary finite multiset over* $\mathcal{O}$ (or *with urelements in* $\mathcal{O}$) if the following conditions hold:

(1) for all $y \in \mathrm{WTC}(x)$, $a \in \mathcal{O}$, and $m, n \in N^+$ if $\langle m, a \rangle \in y$ and $\langle n, a \rangle \in y$, then $m = n$,

(2) for every $y \in \mathrm{WTC}(x) - \{x\}$ there exists a unique natural number $n \in N^+$ such that $\langle n \rangle \in y$,

(3) for all $y, z \in \mathrm{WTC}(x)$ and $m, n \in N^+$ if $\langle m \rangle \in y \in z$ and $\big(y - \{\langle m \rangle\}\big) \cup \{\langle n \rangle\} \in z$, then $m = n$,

(4) $\big\{ \langle n \rangle \mid n \in N^+ \big\} \cap x = \emptyset$.

Let x be a hereditary finite multiset over $\mathcal{O}$, and let $n \in N^+$, $a \in \mathcal{O}$, and $y, z \in \mathrm{WTC}(x)$. By virtue of (1) the formula "$\langle n, a \rangle \in y$" can be interpreted unambiguously such that

exactly n copies of a belong to y,

and by virtue of (2) and (3) the formula "$\langle n \rangle \in y \in z$" can be interpreted unambiguously such that

exactly n copies of $y - \{\langle n \rangle\}$ belong to (are immediately nested in) z.

The above interpretation explains the name "hereditary finite multiset".

Hereditary finite multisets have also an immediate representation by certain multisets understood as usual as functions taking values in natural numbers and described in the following way.

Let $\mathrm{HFM}(\mathcal{O})$ denote the set of hereditary finite multisets over $\mathcal{O}$ and let $\mathrm{MHF}(\mathcal{O})$ denote the set of *finite multisets over* $\mathrm{HFM}(\mathcal{O}) \cup \mathcal{O}$, i.e., those functions $M : \mathrm{HFM}(\mathcal{O}) \cup \mathcal{O} \to N$ valued in the set N of natural numbers with 0 for which the set $\{x \in \mathrm{HFM}(\mathcal{O}) \cup \mathcal{O} \mid M(x) > 0\}$ is a finite set. The following proposition describes the discussed representation.

Proposition 1. *A mapping* $(\cdot)^\S : \mathrm{HFM}(\mathcal{O}) \to \mathrm{MHF}(\mathcal{O})$ *given for* $x \in \mathrm{HFM}(\mathcal{O})$, $z \in \mathrm{HFM}(\mathcal{O}) \cup \mathcal{O}$ *by*

$$(x)^\S(z) = \begin{cases} n & \text{if } \{\langle n \rangle\} \cup z \in x \text{ with } z \in \mathrm{HFM}(\mathcal{O}) \text{ or } \langle n, z \rangle \in x \text{ with } z \in \mathcal{O}, \\ 0 & \text{otherwise,} \end{cases}$$

is a bijection whose inverse $(\cdot)^{-\S} : \mathrm{MHF}(\mathcal{O}) \to \mathrm{HFM}(\mathcal{O})$ *is defined by*

$$(M)^{-\S} = \big\{\langle M(y)\rangle \cup y \mid y \in \mathrm{HFM}(\mathcal{O}) \text{ and } M(y) > 0\big\} \cup \big\{\langle M(a), a\rangle \mid a \in \mathcal{O} \text{ and } M(a) > 0\big\}$$

for $M \in \mathrm{MHF}(\mathcal{O})$.

Proof. The proposition is an immediate consequence of the definitions of $(\cdot)^{\S}$ and $(\cdot)^{-\S}$. □

Thus one defines the sum $+$ and subtraction $\dot{-}$ of hereditary finite multisets by

$$x + y = \big((x)^{\S} + (y)^{\S}\big)^{-\S},$$
$$x \mathbin{\dot{-}} y = \big((x)^{\S} \mathbin{\dot{-}} (y)^{\S}\big)^{-\S},$$

for hereditary finite multisets x, y, where the symbols $+$ and $\dot{-}$ standing on the right hand sides of the above equations denote the sum and the subtraction of multisets, respectively.

We recall that these sum and subtraction are defined pointwise for two multisets M, M' by

$$(M + M')(x) = M(x) + M'(x),$$
$$(M \mathbin{\dot{-}} M')(x) = M(x) \mathbin{\dot{-}} M'(x)$$

for all x and for subtraction of natural numbers given by

$$m \mathbin{\dot{-}} n = \begin{cases} m - n & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1. *The set* $\mathrm{HFM}(\mathcal{O})$ *of hereditary finite multisets over* $\mathcal{O}$ *and their sum* $+$ *defined above form a commutative monoid (with unit* $\varnothing$*) which is partially ordered by* $\leq$ *defined pointwise by*

$$x \leq y \quad \textit{iff} \quad (x)^{\S}(z) \leq (y)^{\S}(z) \textit{ for all } z \in \mathrm{HFM}(\mathcal{O}) \cup \mathcal{O}.$$

We introduce now the notions of a sequential hereditary finite set and sequential hereditary finite multiset.

For a set $\mathcal{O}$ of objects we define inductively a family of sets SHF_i for natural numbers $i \geq 0$ such that

$$\mathrm{SHF}_0 = \varnothing,$$
$$\mathrm{SHF}_{i+1} = \text{the set of nonempty finite subsets of } N^{+} \times (\mathcal{O} \cup \mathrm{SHF}_i \cup \{\varnothing\})^{\clubsuit},$$

where $(\mathcal{O} \cup \mathrm{SHF}_i \cup \{\varnothing\})^{\clubsuit}$ is the set of finite nonempty strings (sequences) of elements of the union $\mathcal{O} \cup \mathrm{SHF}_i \cup \{\varnothing\}$.

The elements of the union $\mathrm{SHF}(\mathcal{O}) = \bigcup\{\mathrm{SHF}_i \mid i \geq 0\} \cup \{\varnothing\}$ are called *sequential hereditary finite sets over* $\mathcal{O}$, or briefly SHF-sets.

For $x \in \mathrm{SHF}(\mathcal{O})$ we define inductively

$$\mathrm{SWTC}(x) = \bigcup\big\{\mathrm{SWTC}(y) \mid y \in \mathrm{SHF}(\mathcal{O}) \text{ and } y \text{ is an element of some string } \Gamma \text{ such that } (n,\Gamma) \in x \text{ for some } n \in N^+\big\} \cup \{x\}.$$

We say that a SHF-set x over $\mathcal{O}$ is a *sequential hereditary finite multiset over* $\mathcal{O}$, or briefly SHF-*multiset*, if for all $y \in \mathrm{SWTC}(x)$, $i, m, n \in N^+$ and $\Gamma \in \big(\mathcal{O} \cup \mathrm{SHF}_i \cup \{\varnothing\}\big)^{\clubsuit}$ if $(m,\Gamma) \in y$ and $(n,\Gamma) \in y$, then $m = n$.

SHF-multisets have an immediate representation by certain multisets as shown below.

Let $\mathrm{SHFM}(\mathcal{O})$ denote the set of SHF-multisets over $\mathcal{O}$ and let $\mathrm{MSHF}(\mathcal{O})$ denote the set of *finite multisets over* $\big(\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})\big)^{\clubsuit}$, i.e., the set of those functions $M : \big(\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})\big)^{\clubsuit} \to N$ for which $\big\{\Gamma \in \big(\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})\big)^{\clubsuit} \mid M(\Gamma) > 0\big\}$ is a finite set, where $\big(\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})\big)^{\clubsuit}$ denotes the set of finite non-empty strings of elements of $\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})$. The following proposition describes the considered representation.

Proposition 2. *A mapping* $(\cdot)^{\#} : \mathrm{SHFM}(\mathcal{O}) \to \mathrm{MSHF}(\mathcal{O})$ *given for* x, Γ *by*

$$(x)^{\#}(\Gamma) = \begin{cases} n & \text{if } (n,\Gamma) \in x, \\ 0 & \text{otherwise,} \end{cases}$$

is a bijection whose inverse $(\cdot)^{-\#} : \mathrm{MSHF}(\mathcal{O}) \to \mathrm{SHFM}(\mathcal{O})$ *is defined by*

$$(M)^{-\#} = \big\{(M(\Gamma),\Gamma) \mid \Gamma \in (\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O}))^{\clubsuit} \text{ and } M(\Gamma) > 0\big\}$$

for $M \in \mathrm{MSHF}(\mathcal{O})$.

Proof. The proposition is an immediate consequence of the definitions of $(\cdot)^{\#}$ and $(\cdot)^{-\#}$. □

Thus one defines the sum $+$ and subtraction $\dot{-}$ of SHF-multisets by

$$x + y = \big((x)^{\#} + (y)^{\#}\big)^{-\#},$$
$$x \dot{-} y = \big((x)^{\#} \dot{-} (y)^{\#}\big)^{-\#}$$

for SHF-multisets x, y, where the symbols $+$ and $\dot{-}$ standing on the right hand sides of the above equations denote the already defined sum and subtraction of multisets, respectively.

Corollary 2. *The set* $\mathrm{SHFM}(\mathcal{O})$ *of SHF-multisets over* $\mathcal{O}$ *and their sum* $+$ *defined above form a commutative monoid* (*with unit* $\varnothing$) *which is partially ordered by* $\leq$ *defined pointwise by*

$$x \leq y \quad \text{iff} \quad (x)^{\#}(\Gamma) \leq (y)^{\#}(\Gamma) \text{ for all } \Gamma \in \big(\mathcal{O} \cup \mathrm{SHFM}(\mathcal{O})\big)^{\clubsuit}.$$

Remark–hint 1. To model systems of processes containing an unbounded number of parallel replicas of some parts, cf. [2], [3], and [4], we introduce new concepts of ∞-*multisets*, *hereditary finite* ∞-*multisets*, and *sequential hereditary finite* ∞-*multisets* by defining them in the same way as multisets, hereditary finite multisets, and sequential hereditary finite multisets, respectively, except we replace the set N of natural numbers with 0 and the set N^+ of natural numbers excluding 0 by the sets $N^\infty = N \cup \{\infty\}$ and $N^{+\infty} = N^+ \cup \{\infty\}$, respectively.

We assume here that

(A_1) $n < \infty$, for all $n \in N$,

(A_2) $m + \infty = \infty = \infty + m$, for all $m \in N^\infty$.

Thus by (A_1) for a ∞-multiset M the equation $M(a) = \infty$ means that there are given an infinite number of copies of an object a.

By using (A_2) one defines the sum $+$ of ∞-multisets, hereditary finite ∞-multisets, and sequential hereditary finite ∞-multisets in an analogous way to the already defined sum of multisets, hereditary finite multisets, and sequential hereditary finite multisets.

We introduce now a unary *operation* **!** *of unbounded replication* which is used in the next section and defined for hereditary finite ∞-multisets in the following way.

For a hereditary finite ∞-multiset x over a set $\mathcal{O}$ of objects we define

$$\begin{aligned}\mathbf{!}(x) = \big\{\langle\infty, a\rangle \mid \langle n, a\rangle \in x \text{ for some } n \in N^{+\infty} \text{ and } a \in \mathcal{O}\big\}\\ \cup\big\{(y - \{\langle n\rangle \mid n \in N^{+\infty}\}) \cup \{\langle\infty\rangle\} \bigm| y \in x\\ \text{and } y - \{\langle n\rangle \mid n \in N^{+\infty}\} \text{ is a hereditary finite } \infty\text{-multiset over } \mathcal{O}\big\}.\end{aligned}$$

The operation **!** has the following properties.

Lemma 1. *For all hereditary finite* ∞*-multisets* x *and* y *over a set* $\mathcal{O}$ *of objects the following equations hold*:

(i) $\mathbf{!}(x + y) = \mathbf{!}(x) + \mathbf{!}(y)$,

(ii) $\mathbf{!}(x) = \mathbf{!}(x) + x$,

(iii) $\mathbf{!}(\mathbf{!}(x)) = \mathbf{!}(x)$,

(iv) $\mathbf{!}(\emptyset) = \emptyset$.

Proof. The lemma is an immediate consequence of the definitions of **!** and the sum of hereditary finite multisets. □

A counterpart of the operation of unbounded replication defined for sequential hereditary finite ∞-multisets in a similar way has also the properties described in Lemma 1.

3. Interpretation of some reducts of Ambient Calculus and Brane Calculi in certain classes of hereditary finite multisets and sequential hereditary finite multisets

In this section we present the mentioned models of some reducts of considered calculi by defining the valuations of appropriate expressions of the language of some reduct of Ambient Calculus [1] and the language of some reduct of Mate/Bud/Drip Brane Calculus [2], [3] in a certain class of hereditary finite multisets and in a certain class of sequential hereditary finite multisets, respectively. The valuations are defined inductively on the construction of expressions in a similar way as terms and formulas of predicate calculi are valuated in relational systems. Then we show a completeness of the valuations by meaning that two expressions of the considered languages are structurally congruent (equivalent) if their valuations are equal.

We begin with a presentation of basic syntactic features of considered reducts of discussed calculi.

The syntax of considered reduct of Ambient Calculus is determined by three syntactic categories: a category of names *Names*, a category of capabilities *Capabilities*, and a category of processes *Processes*, where these categories are sets of expressions specified by using Bachus–Naur form (BNF) notation in the following way.

Capabilities:

$$M ::= in\,\boldsymbol{n} \mid out\,\boldsymbol{n} \mid open\,\boldsymbol{n}$$

Processes:

$$P, Q ::= 0 \mid P|Q \mid \boldsymbol{n}[P] \mid M.P,$$

where $\boldsymbol{n}$ is a metavariable ranging over expressions belonging to the category of names which is not specified here.

The syntax of considered reduct of Mate/Bud/Drip Brane Calculus is determined by four syntactic categories: a category of names *Names*, a category of actions *Actions*, a category of branes *Branes*, and a category of systems *Systems*, where these categories are sets of expressions specified by using BNF notation in the following way.

Branes:

$$\sigma, \tau ::= 0 \mid \sigma|\tau \mid a.\sigma$$

Actions:

$$a ::= mate_{\boldsymbol{n}} \mid mate^{\perp}_{\boldsymbol{n}} \mid bud_{n} \mid bud^{\perp}_{\boldsymbol{n}}(\sigma) \mid drip(\sigma)$$

Systems:

$$P, Q ::= \diamond \mid P \circ Q \mid \sigma(\!|P|\!),$$

where $\boldsymbol{n}$ is a metavariable ranging over expressions belonging to the category of names which is not specified here.

For the reduct of Ambient Calculus given by the syntactic categories *Names*, *Capabilities*, and *Processes* described above, we introduce a *valuation* of expressions of this reduct in the set $\mathrm{HFM}(\dot{\mathcal{O}})$ of hereditary finite multisets over $\dot{\mathcal{O}} = Names \cup Capabilities$. The valuation is an inductively defined function $[\![\cdot]\!] : Processes \to \mathrm{HFM}(\dot{\mathcal{O}})$ by the clauses:

- $[\![0]\!] = \varnothing$;
- $[\![\boldsymbol{n}[P]]\!] = \{[\![P]\!] \cup \{\langle 1\rangle, \langle 1, \boldsymbol{n}\rangle\}\}$;
- $[\![M.P]\!] = \{[\![P]\!] \cup \{\langle 1\rangle, \langle 1, M\rangle\}\}$;
- $[\![P|Q]\!] = [\![P]\!] + [\![Q]\!]$,

where $+$ is the sum of hereditary finite multisets defined in Section 2.

The following theorem shows a completeness of the valuation $[\![\cdot]\!]$ with respect to a structural congruence $\asymp$ defined for expressions belonging to the syntactic category *Processes* in the same way as the structural congruence $\equiv$ of processes is defined in [4] except we exclude these clauses defining $\equiv$ which concern the operations not specified in the construction of the category *Processes* like for instance the operation ! of replication.

Theorem 1. *For all expressions P, Q belonging to the syntactic category Processes if the equation $[\![P]\!] = [\![Q]\!]$ holds, then P is structurally congruent to Q, i.e., $P \asymp Q$.*

Proof. For $x \in \mathrm{HFM}(\mathcal{O})$ we define the *depth* of x to be the smallest number in $\{i \mid i \geq 0 \text{ and } x \in \mathrm{HF}_i\}$. One constructs a monoid homomorphism

$$h : \{[\![P]\!] \mid P \in \mathit{Processes}\} \to \{(P/\asymp) \mid P \in \mathit{Processes}\}$$

by induction on depth of elements of $\{[\![P]\!] \mid P \in \mathit{Processes}\}$, where $(P/\asymp)$ denotes the set of these $Q \in \mathit{Processes}$ for which $Q \asymp P$. Then one proves by induction on the construction of elements of the category *Processes* that

$$P \in h([\![P]\!]) \quad \text{for all} \quad P \in \mathit{Processes}.$$

Hence for P, Q the equation $[\![P]\!] = [\![Q]\!]$ implies that

$$(P/\asymp) = h([\![P]\!]) = h([\![Q]\!]) = (Q/\asymp).$$ □

For the reduct of Mate/Bud/Drip Brane Calculus given by the syntactic categories *Names*, *Actions*, *Branes*, *Systems* described above, we introduce a *valuation* of expressions of this reduct. The valuation is given by three functions

$$\begin{aligned} [\![\cdot]\!]^{\boldsymbol{a}} &: \mathit{Actions} \to (\mathrm{SHFM}(\ddot{\mathcal{O}}) \cup \ddot{\mathcal{O}})^{\clubsuit}, \\ [\![\cdot]\!]^{\boldsymbol{b}} &: \mathit{Branes} \to \mathrm{SHFM}(\ddot{\mathcal{O}}), \\ [\![\cdot]\!]^{\boldsymbol{s}} &: \mathit{Systems} \to \mathrm{SHFM}(\ddot{\mathcal{O}}) \end{aligned}$$

for the set $\mathrm{SHFM}(\ddot{\mathcal{O}})$ of sequential hereditary finite multisets over

$$\ddot{\mathcal{O}} = \{\delta_{\boldsymbol{n}} \mid \delta \in \{\mathit{mate}, \mathit{mate}^{\perp}, \mathit{bud}, \mathit{bud}^{\perp}\} \text{ and } \boldsymbol{n} \in \mathit{Names}\} \cup \{\mathit{drip}, \mathit{mbrn}\}.$$

The functions $[\![\cdot]\!]^{\boldsymbol{x}}$ $(\boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}\})$ are defined by the clauses:

– $[\![0]\!]^{\boldsymbol{b}} = \varnothing$ and $[\![\diamond]\!]^{\boldsymbol{s}} = \varnothing$,

– for an action a of one of the forms $mate_{\boldsymbol{n}}$, $mate_{\boldsymbol{n}}^{\perp}$, $bud_{\boldsymbol{n}}$ ($\boldsymbol{n} \in Names$)

$$[\![a]\!]^{\boldsymbol{a}} = a, \text{ i.e., } [\![a]\!]^{\boldsymbol{a}} \text{ is a one-element string } a,$$

– for an action a of one of the forms $bud_{\boldsymbol{n}}^{\perp}(\sigma)$, $drip(\sigma)$ ($\boldsymbol{n} \in Names$, $\sigma \in Branes$)

$$[\![a]\!]^{\boldsymbol{a}} = \begin{cases} bud_{\boldsymbol{n}}^{\perp} \frown [\![\sigma]\!]^{\boldsymbol{b}} & \text{if } a \text{ is } bud_{\boldsymbol{n}}^{\perp}(\sigma), \\ drip \frown [\![\sigma]\!]^{\boldsymbol{b}} & \text{if } a \text{ is } drip(\sigma), \end{cases}$$

where $\frown$ denotes concatenation (juxtaposition) of finite strings,

– for an action a and a brane σ

$$[\![a.\sigma]\!]^{\boldsymbol{b}} = \{(1, [\![a]\!]^{\boldsymbol{a}} \frown [\![\sigma]\!]^{\boldsymbol{b}})\},$$

– for two branes σ, τ

$$[\![\sigma|\tau]\!]^{\boldsymbol{b}} = [\![\sigma]\!]^{\boldsymbol{b}} + [\![\tau]\!]^{\boldsymbol{b}},$$

where $+$ is the sum of SHF-multisets defined in Section 2,

– for a brane σ and a system P

$$[\![\sigma(\!|P|\!)]\!]^{\boldsymbol{s}} = \begin{cases} \{(1, mbrn \frown x)\} & \text{if at least one of } [\![\sigma]\!]^{\boldsymbol{b}} \\ & \text{or } [\![P]\!]^{\boldsymbol{s}} \text{ is non-empty}, \\ \varnothing & \text{otherwise}, \end{cases}$$

where $x = [\![P]\!]^{\boldsymbol{s}} \cup [\![\sigma]\!]^{\boldsymbol{b}}$,

– $[\![P \circ Q]\!]^{\boldsymbol{s}} = [\![P]\!]^{\boldsymbol{s}} + [\![Q]\!]^{\boldsymbol{s}}$ for systems P, Q.

The following theorem shows a completeness of the valuation $([\![\cdot]\!]^{\boldsymbol{x}} \mid \boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}\})$ with respect to structural congruences $\asymp_{\boldsymbol{b}}$, $\asymp_{\boldsymbol{s}}$ defined for expressions belonging to the syntactic categories *Branes* and *Systems*, respectively, in the same way as the structural congruence $\equiv$ for branes and systems is defined in [2] or [3] except we exclude the clauses defining $\equiv$ which concern the operators not specified in the definitions of *Branes* and *Systems*.

Theorem 2. *The following two propositions hold*:

(A) *for all expressions σ, τ belonging to the syntactic category of branes if $[\![\sigma]\!]^{\boldsymbol{b}} = [\![\tau]\!]^{\boldsymbol{b}}$, then $\sigma \asymp_{\boldsymbol{b}} \tau$,*

(B) *for all expressions P, Q belonging to the syntactic category of systems if $[\![P]\!]^{\boldsymbol{s}} = [\![Q]\!]^{\boldsymbol{s}}$, then $P \asymp_{\boldsymbol{s}} Q$.*

Proof. The proof is similar to that of Theorem 1. □

We describe now a useful construction of weak membrane systems from SHF-multisets belonging to $\big\{ [\![P]\!]^{\boldsymbol{s}} \mid P \in \mathit{Systems} \big\}$ which is a counterpart of the construction of weak membrane systems $\mathit{expans}(x)$ from hereditary finite multisets x, see the Appendix. For unexplained terms and notation see that Appendix.

For an SHF-multiset $x = [\![P]\!]^{\boldsymbol{s}}$ for some $P \in \mathit{Systems}$ we introduce the following notation and new notions.

We define

$$\mathrm{MBRN}(x) = \big\{ (n, \mathit{mbrn} \frown z) \mid n \in N^+, z \in \mathrm{SWTC}(x) \big\} \cup \{x\}.$$

For $y, z \in \mathrm{MBRN}(x)$ we define that y is *immediately nested in* z, briefly $y \prec z$, in the following way:

$$\begin{aligned} y \prec z \quad \text{iff} \quad & y \in x \text{ with } z = x \text{ or } y \in \dot{z} \text{ for that } \dot{z} \\ & \text{for which } z = (n, \mathit{mbrn} \frown \dot{z}) \text{ for some } n \in N^+. \end{aligned}$$

Then we define the *multigraf* $\dot{G}_x$ *associated to* x by

- the set $\dot{V}_x$ of vertices of $\dot{G}_x$ is the set $\mathrm{MBRN}(x)$,
- the set $\dot{E}_x$ of arrows is the set

$$\begin{aligned} \big\{ & (y, i, z) \in \dot{V}_x \times N^+ \times \dot{V}_x \mid y \succ z \text{ and } 1 \leq i \leq n \\ & \text{for that } n \text{ and that } \dot{z} \text{ for which } z = (n, \mathit{mbrn} \frown \dot{z}) \big\} \end{aligned}$$

- the source and the target functions of $\dot{G}_x$ are the projections on the first and the third component.

The tree structure $\mathit{Path}(\dot{G}_x, x)$ of paths from x in $\dot{G}_x$ is the underlying tree structure of the claimed weak membrane system $\mathit{exbrn}(x)$ constructed from x, where the object distribution $M_{\mathit{exbrn}(x)}$ is defined in the following way.

For a nonempty path π belonging to the underlying set of tree structure $\mathit{Path}(\dot{G}_x, x)$ such that $(n, \mathit{mbrn} \frown z)$ is the target of the last element of π we define $M_{\mathit{exbrn}(x)}(\pi)$ to be that multiset which is the restriction of $\big(z \cap \big\{ [\![\sigma]\!]^{\boldsymbol{b}} \mid \sigma \in \mathit{Branes} \big\}\big)^{\#}$ to the set $\big\{ [\![\sigma]\!]^{\boldsymbol{b}} \mid \sigma \in \mathit{Branes} \big\}$. For empty path π we define $M_{\mathit{exbrn}(x)}(\pi)$ to be the restriction of $(\varnothing)^{\#}$ to $\big\{ [\![\sigma]\!]^{\boldsymbol{b}} \mid \sigma \in \mathit{Branes} \big\}$.

4. Conclusions

One can treat the valuation $[\![\cdot]\!]$ together with the set $\big\{ [\![P]\!] \mid P \in \mathit{Processes} \big\}$ as a mathematical model of the considered restriction of Ambient Calculus and the valuation $\big([\![\cdot]\!]^{\boldsymbol{x}} \mid \boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}\} \big)$ together with the sets $\big\{ [\![a]\!]^{\boldsymbol{a}} \mid a \in \mathit{Actions} \big\}$, $\big\{ [\![\sigma]\!]^{\boldsymbol{b}} \mid \sigma \in \mathit{Branes} \big\}$, $\big\{ [\![P]\!]^{\boldsymbol{s}} \mid P \in \mathit{Systems} \big\}$ as a mathematical model of the considered reduct of Mate/Bud/Drip Brane Calculus. These mathematical models besides providing a denotation of expressions of the languages of considered calculi via valuations have also some practical interpretation. Namely, one can identify processes P (i.e., expressions belonging to *Processes*), modulo structural congruence, with membrane systems $\mathit{expans}(\llbracket P \rrbracket)$ (see Appendix for the weak membrane system $\mathit{expans}(x)$) and then one can investigate (the considered reduct of) Ambient Calculus in terms of membrane computing by using evolution rules related to reduction relation of processes in Ambient Calculus, cf. [9].

A similar identification of expressions belonging to the syntactic category of *Systems* with weak membrane systems determined by the construction of $\mathit{exbrn}(x)$ describes the links between membrane computing and Brane Calculi, already pointed out and discussed in [5], in a mathematical way on semantical (model theoretical) level.

Remark–hint 2. The systems or processes containing an unbounded number of parallel replicas of their parts are described in terms of Ambient Calculus and Brane Calculi by using expressions of the forms $!P$, $!\sigma$, and structural congruences satisfying among others the conditions similar to equations in Lemma 1, see [2], [3], [4].

One can extend the valuation $\llbracket \cdot \rrbracket$ of expressions in the category *Processes* to expressions $!P$ by defining

$$\llbracket !P \rrbracket = \mathbf{!}(\llbracket P \rrbracket),$$

where the extended valuation $\llbracket \cdot \rrbracket$ is a mapping into the class of hereditary finite ∞-multisets over $\dot{\mathcal{O}}$ and $\mathbf{!}$ is the operation of unbounded replication introduced in Section 2.

Thus by Lemma 1 one obtains a mathematical model of the reduct of Ambient Calculus comprising unbounded replication, where the model is determined by the above defined extension of the valuation $\llbracket \cdot \rrbracket$. One can extend the valuation $\big(\llbracket \cdot \rrbracket^{\boldsymbol{x}} \mid \boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}\}\big)$ in a similar way to obtain mathematical models of the reducts of Brane Calculi comprising unbounded replication.

Appendix

We recall the notions of a tree structure and of an abstract membrane system as have been introduced in [9].

For a vertex v of a directed nonempty graph G we define a *route from v in G* to be a finite string $v_1 \dots v_n$ with $n \geq 1$ of vertices of G such that $v_1 = v$ and if $n > 1$, then $(v_i, v_{i+1}) \in E$ for all i with $1 \leq i < n$, where E is the set of edges of G and $E \subseteq V \times V$ for the set V of vertices of G.

A [*finite*] *tree structure* is a unary algebra $\mathcal{T} = (T_{\mathcal{T}}, \eta_{\mathcal{T}}, r_{\mathcal{T}})$, where $T_{\mathcal{T}}$ is a [finite] set, called the *underlying set* of $\mathcal{T}$, and unary operation $\eta_{\mathcal{T}} : T_{\mathcal{T}} \to T_{\mathcal{T}}$ with constant $r_{\mathcal{T}} \in T_{\mathcal{T}}$ are such that the *underlying graph* $G_{\mathcal{T}} = (T_{\mathcal{T}}, E_{\mathcal{T}})$ of $\mathcal{T}$ with the set $T_{\mathcal{T}}$ of vertices and the set $E_{\mathcal{T}} = \big\{(\eta_{\mathcal{T}}(t), t) \mid t \in T_{\mathcal{T}} - \{r_{\mathcal{T}}\}\big\}$ of edges of $G_{\mathcal{T}}$ is a tree with the root $r_{\mathcal{T}}$[1] and $\eta_{\mathcal{T}}(r_{\mathcal{T}}) = r_{\mathcal{T}}$. Unary operation $\eta_{\mathcal{T}}$ is called *immediate nesting*

[1] $G_{\mathcal{T}}$ is a graph such that for every $t \in T_{\mathcal{T}}$ there exists a unique route from $r_{\mathcal{T}}$ in $G_{\mathcal{T}}$ whose last element is t.

operation of $\mathcal{T}$ and $r_{\mathcal{T}}$ is called the *root* of $\mathcal{T}$. The notion of a tree structure is a generalization of the notion of a membrane structure introduced in [10].

We recall that a *directed multigraph* G is given by its *set* V *of vertices*, its *set* E *of arrows*, and two functions $s, t : E \to V$, called *source* and *target functions* of G, respectively, where the values $s(e)$ and $t(e)$ are called the *source* and the *target* of $e \in E$, respectively.

By a *path from a vertex* v in a directed multigraph G we mean a finite string π, may be empty, of arrows of G such that for a nonempty π the source of the first element of π is v and for π of the form $e_1 \dots e_n$ with $n > 1$ we impose $s(e_i) = t(e_{i-1})$ for every i with $1 \leq i \leq n$.

For a directed multigraph G and its vertex v we define a tree structure $\mathrm{Path}(G, v)$ such that

- the underlying set of $\mathrm{Path}(G, v)$ is the set of all paths from v in G,
- the immediate nesting operation η of $\mathrm{Path}(G, v)$ is given by

$$\eta(\pi) = \begin{cases} e_1 \dots e_{n-1} & \text{if } \pi \text{ is a string } e_1 \dots e_n \text{ with } n > 1, \\ \text{empty path} & \text{if } \pi \text{ is empty path or one element path } e_1, \end{cases}$$

- the root of $\mathrm{Path}(G, v)$ is empty path.

An *abstract membrane system* $\mathcal{S}$ is defined to be a tree structure $\mathcal{T}_{\mathcal{S}} = (T_{\mathcal{S}}, \eta_{\mathcal{S}}, r_{\mathcal{S}})$ equipped with three mappings $\ell_{\mathcal{S}} : T_{\mathcal{S}} \to L_{\mathcal{S}}$, $e_{\mathcal{S}} : T_{\mathcal{S}} \to \{-, +, 0\}$ and $M_{\mathcal{S}} : T_{\mathcal{S}} \to N^{O_{\mathcal{S}}}$, where $L_{\mathcal{S}}$ is the *set of labels* of $\mathcal{S}$ and $N^{O_{\mathcal{S}}}$ is the set of multisets over the *set* $O_{\mathcal{S}}$ *of objects* of $\mathcal{S}$, i.e., the set of functions defined on set $O_{\mathcal{S}}$ and valued in the set N of natural numbers (thus, the characteristic functions of subsets of $O_{\mathcal{S}}$ are among multisets over $O_{\mathcal{S}}$). The tree structure $\mathcal{T}_{\mathcal{S}}$ is called the *underlying tree structure* of $\mathcal{S}$ and the elements of $\mathcal{T}_{\mathcal{S}}$ are called *membranes* of $\mathcal{S}$, where the root of $\mathcal{T}_{\mathcal{S}}$ is called the *skin* of $\mathcal{S}$ and a membrane $m \in T_{\mathcal{S}}$ is called an *elementary membrane* if $\{m' \mid \eta_{\mathcal{S}}(m') = m\}$ is empty. The functions $\ell_{\mathcal{S}}, e_{\mathcal{S}}, M_{\mathcal{S}}$ are called *labelling function*, *electric charge function*, and *object distribution function* of $\mathcal{S}$, respectively. For a membrane $m \in T_{\mathcal{S}}$ the values $\ell_{\mathcal{S}}(m)$, $e_{\mathcal{S}}(m)$, $M_{\mathcal{S}}(m)$ are the label of m, the electric charge of m, the multiset of objects contained in the region of m, respectively.

To make the discussed definitions more concise and the constructions sufficiently general, we define a [*finite*] *weak membrane system* S to be a [finite] tree structure $\mathcal{T}_S = (T_S, \eta_S, r_S)$ equipped with a function $\mathcal{M}_S : T_S \to N^{\mathcal{O}_S}$ valued in the set $N^{\mathcal{O}_S}$ of multisets over $\mathcal{O}_S$. For a weak membrane system S the tree structure $\mathcal{T}_S$, the function $\mathcal{M}_S$, and the set $\mathcal{O}_S$ are called the *underlying tree structure of* S, the *object distribution function of* S, and the *set of objects of* S, respectively.

Since a tree structure can be identified with its underlying graph, which is a tree, one may briefly say that a weak membrane system is a tree whose vertices are labelled by multisets over some set of objects.

Remark 1. The abstract membrane systems can be treated as a special case of weak membrane systems which is defined by imposing that for a weak membrane system S there is given a set L of labels of membranes such that the following conditions hold:

a) the product $\{-, 0, +\} \times L$ is a subset of $\mathcal{O}_S$,

b) for every $m \in T_S$ there exists a unique $p \in \{-, 0, +\} \times L$ such that $\mathcal{M}_S(m)(p) > 0$,

c) for all $m \in T_S$, $p \in \{-, 0, +\} \times L$, if $\mathcal{M}_S(m)(p) > 0$, then $\mathcal{M}_S(m)(p) = 1$.

Thus the proper objects of S are elements of the difference $\mathcal{O}_S - \big(\{-, 0, +\} \times L\big)$, the electric charge $e(m) = \alpha$ and the label $l(m) = l$ of a membrane m are such that $\mathcal{M}_S(m)\big((\alpha, l)\big) = 1$.

We recall the construction of *expans*(x) described in [7].

For a hereditary finite multiset x over $\mathcal{O}$ we define its *ordered multigraph* G_x such that the set V_x of vertices of G_x is the set $\mathrm{WTC}(x)$, the set E_x of arrows of G_x is the set of ordered triples $(y, i, z) \in \mathrm{WTC}(x) \times N^+ \times \mathrm{WTC}(x)$ with $z \in y$ and $1 \leq i \leq k$ for that unique k for which $\langle k \rangle \in z$. The source and target functions of G_x are the projections on the first and third component, respectively.

For a hereditary finite multiset x over $\mathcal{O}$ we define its *expansion expans*(x) to be a weak membrane system whose underlying tree structure is $\mathrm{Path}(G_x, x)$ for the ordered multigraph G_x of x, the set of objects of *expans*(x) is the set $\mathcal{O}$, and the object distribution function $\mathcal{M}_{expans(x)}$ of *expans*(x) is defined by

$$\mathcal{M}_{expans(x)}(\pi)(a) = \begin{cases} n & \text{if the urelement } \langle n, a \rangle \text{ belongs to the target} \\ & \text{of the last element of nonempty } \pi \\ & \text{or } \langle n, a \rangle \in x \text{ in the case of } \pi \text{ being empty,} \\ 0 & \text{otherwise,} \end{cases}$$

for every path π from x in G_x and every $a \in \mathcal{O}$.

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