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Sugeno Measures on the Code Space

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Abstract. One considers a finite set X (*the alphabet*), the code space X^∞ of all sequences formed with elements (*letters*) from the alphabet X and $\mathcal{B}$ the Borel sets of X^∞. A concrete representation of all Sugeno measures on $\mathcal{B}$ is given. This representation is, heuristically speaking, matricial, being inspired by the concrete representation of all probabilities on $\mathcal{B}$.

Key-words: Borel sets; Code space; Monotone measure; Sugeno measure.

1. Introduction

Classical measure theory is based upon the concept of additivity (or, which is more, countable additivity). Recently, new necessities (theoretical and practical) imposed the study of monotone and (possibly) non additive measures, which we shall call generalized measures. Now, these measures play an important role in describing a lot of phenomena and are intensively studied. For instance, one can observe that superadditive measures indicate a cooperative action or synergy between the measured sets, while subadditive measures indicate inhibitory effects, lack of cooperation or incompatibility between the measured sets. Additive measures can express non interaction, indifference.

It is a general opinion that the most important generalized measures are the $\lambda-$measures, introduced by M. Sugeno in his doctoral thesis [12] under the name of *fuzzy measures*. Today, normalized $\lambda-$measures are generally called *Sugeno measures* ([14]). Let us notice an important result due to Z. Wang ([13]) (see also [9], [14]) which states that $\lambda-$measures can be obtained from classical measures via composition with a special increasing function, thus becoming quasi$-$measures.

Aside from the mathematical interest of the Sugeno measures, their impact in other domains of study should also be noticed. For instance, Sugeno measures on $\mathcal{P}(\mathbb{N})$ are special cases of belief measures or plausibility measures (see [11], [14]). The problem of the existence of $\lambda-$measures on $\mathcal{P}(\mathbb{N})$ with preassigned values was recently completely solved ([5]).

In the present paper, Sugeno measures on the code space will be considered. More precise, let X be a finite set (*the alphabet*) whose elements are called *letters*. Let X^∞ be the set of all sequences with terms from X. The space X^∞ is called the *code space*, its elements being called *codes*. Viewing X^∞ as the countable product of countable many copies of the discrete space X, let $\mathcal{B}$ be the class of Borel sets of X^∞. The space X^∞ and its Borel sets $\mathcal{B}$ are of major importance in many branches of mathematics, such as Dynamical Systems Theory (see, *e.g.* [2], [10]), Theory of Fractals (see, *e.g.* [1], [7], [10]) and the Theory of Random Sequences (see, *e.g.* [3], [4]).

The present paper is dedicated to the study of the Sugeno measures on $\mathcal{B}$. More precisely, a concrete representation of these measures in a 'matricial' form will be given.

The main theoretical tool used for this purpose is the monograph of Wang and Klir ([14]). For measure theory notions, one can use [6] and [8].

2. Preliminary Facts

Throughout this paper $\mathbb{N} = \{1, 2, ..., n, ...\}$ and $\mathbb{R}_+ = [0, \infty)$. All sequences $(x_n)_n$ will be indexed with $\mathbb{N}$. When writing $(x_n)_n \subset A$, we mean that $x_n \in A$ for any n.

A. We shall consider a non empty set T with $\mathcal{P}(T) = \{A \,|A \subset T\}$. Let $\mathcal{C} \subset \mathcal{P}(T)$ be an algebra (of sets). A function $\mu : \mathcal{C} \to \mathbb{R}_+$ is called a *monotone measure* if $\mu(\emptyset) = 0$, μ is increasing *i.e.* if A, B are in $\mathcal{C}$ and $A \subset B$, one has $\mu(A) \le \mu(B)$ and μ is not null (*i.e.* $\mu(T) = a > 0$). Let $\lambda \in \left(-\frac{1}{a}, \infty\right)$.

We shall say that μ *satisfies the* λ *– rule* if, for any A, B in $\mathcal{C}$ such that $A \cap B = \emptyset$, one has:

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\,\mu(B)$$

(of course, *0 – rule* means (finite) additivity).

More particular, we shall say that μ *is a* λ *– measure* if, for any disjoint sequence $(E_n)_n \subset \mathcal{C}$ such that $E = \bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{C}$, one has

$$\mu(E) = \begin{cases} \sum\limits_{n=1}^{\infty} \mu(E_n) & \text{if } \lambda = 0 \\ \dfrac{1}{\lambda}\left(\prod\limits_{n=1}^{\infty}(1 + \lambda\mu(E_n)) - 1\right) & \text{if } \lambda \ne 0 \end{cases}$$

(of course, if μ is a 0 – measure, it is σ – additive). One can see that, if μ is a λ – measure, it follows that μ satisfies the λ – rule (the converse assertion being false).

If $\mu(T) = a = 1$ and μ is a λ – measure, we say that μ *is a* λ – *Sugeno measure*. In case there exists $\lambda \in (-1, \infty)$ such that μ is a λ – Sugeno measure, we say that μ is a *Sugeno measure*.

Let $\mathcal{M} = \{\mu : \mathcal{C} \to \mathbb{R}_+ | \mu(T) = 1 \text{ and } \mu \text{ is } \sigma - \text{ additive}\}$. For any $0 \neq \lambda \in (-1, \infty)$, we have the strictly increasing bijection $h_\lambda : [0, 1] \to [0, 1]$, given via

$$h_\lambda(x) = \frac{(\lambda + 1)^x - 1}{\lambda}.$$

For any $\lambda \in (-1, \infty) \setminus \{0\}$, we write $\mathcal{S}_\lambda = \{m : \mathcal{C} \to \mathbb{R}_+ | m \text{ is a } \lambda - \text{ Sugeno measure}\}$. A classical theorem of Wang ([13]) (see also [14]) says that there exists a bijection $V_\lambda : \mathcal{M} \to \mathcal{S}_\lambda$ acting via $V_\lambda(\mu) = m$, where $m(A) = h_\lambda(\mu(A))$, for any $A \in \mathcal{C}$.

B. Now, some basic facts about the *code space*.

Let $2 \leq p \in \mathbb{N}$. We shall consider p distinct elements $x_1, x_2, ..., x_p$ (called *letters*) and write $X = \{x_1, x_2, ..., x_p\}$ (call X the *alphabet*). In most cases one takes $X = \{0, 1, ..., p - 1\}$.

We can introduce $X^* =$ *the free monoid generated by* X namely X^* is formed with all *words* of the form $x = u_1 u_2 ... u_n$ (where $u_i \in X$) with length $l(x) = n$ and we consider also, *the empty word* $e \in X^*$, with length $l(e) = 0$. For x, y in X^*, we write $x < y$ to denote the fact that x *is a prefix of* y and this means: either $x = e$, or $x = u_1 u_2 ... u_n \neq e$ and $y = v_1 v_2 ... v_m$ with $m \geq n$, such that $v_1 = u_1$, $v_2 = u_2, ..., v_n = u_n$. We accept that $X \subset X^*$, *i.e.* $x \in X$ may be viewed in X^*.

Now we introduce

$$X^\infty = X^\mathbb{N} = \{f : \mathbb{N} \to X\}.$$

Namely, an element $f \in X^\infty$ will be considered as follows:

$$f \equiv x \equiv u_1 u_2 ... u_n ...,$$

where $u_n = f(n)$ for any $n \in \mathbb{N}$. So, the elements in X^∞ are sequences of elements in X. We call the elements in X^∞ *codes* and X^∞ is called the *code space*.

For any $x \in X^*$, we can form the set xX^∞. Namely, if $x = e$, define $eX^\infty = X^\infty$ and, for $x = u_1 u_2 ... u_n$, xX^∞ is formed with all sequences $v = v_1 v_2 ... v_m ...$ such that $v_1 = u_1, v_2 = u_2, ..., v_n = u_n$. Clearly, one has $xX^\infty \subset yX^\infty \Leftrightarrow y < x$.

Considering the (metrizable and compact) topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is the discrete topology, write $(X_n, \mathcal{T}_n) = (X, \mathcal{T})$ for any $n \in \mathbb{N}$. Then $X^\infty = \prod_{n=1}^{\infty} X_n$ and we can consider on X^∞ the topology $\mathcal{U} =$ the product topology of the topologies $\mathcal{T}_n$. Then $(X^\infty, \mathcal{U})$ is a metrizable and compact topological space. This space is second countable, namely it has the countable base $\mathcal{P} = \{xX^\infty | x \in X^*\}$, formed with sets which are both open and compact. The Borel sets of $(X^\infty, \mathcal{U})$ will be denoted by $\mathcal{B}$.

Because $\mathcal{P}$ is a generalized semiring which generates $\mathcal{B}$, it will be sufficient for characterizing σ – additive or Sugeno measures on $\mathcal{B}$ (*i.e.* such measures which coincide on $\mathcal{P}$ must be equal and the values on $\mathcal{P}$ determine all the values).

Caution: Within this framework, we shall denote from now on (see **A.**):

$$\mathcal{M} = \{\mu : \mathcal{B} \to \mathbb{R}_+ | \, \mu(X^\infty) = 1 \text{ and } \mu \text{ is } \sigma - \text{ additive}\}.$$

For further purposes, write $U_p = \{1, 2, ..., p\}$.

A concrete description of the elements in $\mathcal{M}$ can be obtained as follows. Call *0 – distribution* (on X^∞) a sequence $(D(n))_n$, where:

$$\begin{array}{rcl} D(1) & = & (a(1), a(2), ..., a(p)) = (a(i))_{1 \le i \le p} \\ D(2) & = & (a(i,j))_{1 \le i \le p, 1 \le j \le p} \\ - - - & - & - - - - - - - - - \\ D(n) & = & (a(i_1, i_2, ..., i_n))_{1 \le i_k \le p} \quad \text{for} \quad k = 1, 2, ..., n \end{array}$$

($D(n)$ has p^n elements) such that:

a) For any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$, one has $a(i_1, i_2, ..., i_n) \ge 0$;

b) $\sum\limits_{i=1}^{p} a(i) = 1$;

c) For any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$, $a(i_1, i_2, ..., i_n) = \sum\limits_{i=1}^{p} a(i_1, i_2, ..., i_n, i)$.

The most popular 0 – distribution is the *Bernoulli distribution*, which is defined using only the initial distribution $D(1)$: the transition distributions $D(n)$, for $n \ge 2$, are defined via $a(i_1, i_2, ..., i_n) = a(i_1)\, a(i_2) ... a(i_n)$. If $a(i) = \frac{1}{p}$, for $i = 1, 2, ..., p$, we have the *uniform distribution*.

Denote by $\mathcal{D}_0$ the set of all 0 – distributions. The announced concrete description is given by the bijection $T_0 : \mathcal{D}_0 \to \mathcal{M}$, defined as follows: $T_0\left((D(n))_n\right) = \mu$, where $\mu : \mathcal{B} \to \mathbb{R}_+$ acts, for any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$, via

$$\mu(x_{i_1} x_{i_2} ... x_{i_n} X^\infty) = a(i_1, i_2, ..., i_n)$$

$\left(\text{here } D(n) = (a(i_1, i_2, ..., i_n))_{1 \le i_k \le p}\right)$.

Caution. In the following paragraph, we shall denote, for any $\lambda \in (-1, \infty) \setminus \{0\}$

$$\mathcal{S}_\lambda = \{\mu : \mathcal{B} \to \mathbb{R}_+ \, | \mu \text{ is a } \lambda - \text{ Sugeno measure}\}$$

(see A.) and

$$\mathcal{S} = \bigcup_{\lambda \in (-1,\infty) \setminus \{0\}} \mathcal{S}_\lambda.$$

3. Results

A. We will begin with an introductory part, containing results which will be used at the end of the paper.

Lemma 3.1. *Let $\mathcal{C} \subset \mathcal{P}(T)$ be an algebra and $\mu : \mathcal{C} \to \mathbb{R}_+$ a monotone measure with $\mu(T) = a > 0$. Assume μ satisfies the λ – rule, where $\lambda \in \left(-\frac{1}{a}, \infty\right)$. Let A, B in $\mathcal{C}$.*

1. *If $\mu(B) = 0$, one has $\mu(A \cup B) = \mu(A \backslash B) = \mu(A \Delta B) = \mu(A)$, i.e. μ is null – additive.*
2. *If $\mu(A) = a$, one has $\mu(B) = \mu(B \cap A)$ (in particular $\mu(B) = 0$, whenever $B \cap A = \emptyset$).*
3. *If $0 < \mu(A) < a$, it follows that $0 < \mu(T \backslash A) < a$.*

Proof.

1. Because $A \cup B = A \cup (B \backslash A)$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B \backslash A) + \lambda \mu(A) \mu(B \backslash A)$$

and $\mu(B \backslash A) = 0$.

Consequently, because $A = (A \backslash (A \cap B)) \cup (A \cap B)$, we have

$$\mu(A) = \mu(A \backslash (A \cap B)) = \mu(A \backslash B).$$

It follows, because $A \Delta B = (A \cup B) \backslash (A \cap B)$, that $\mu(A \Delta B) = \mu(A \cup B) = \mu(A)$.

2. Assume first $B \cap A = \emptyset$. Then

$$a = \mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A) \mu(B) = a + \mu(B) + \lambda a \mu(B),$$

hence $\mu(B)(1 + \lambda a) = 0$. We get $\mu(B) = 0$, because $1 + \lambda a > 0$.

In the general case, $B = (B \cap A) \cup (B \backslash A)$ with $(B \backslash A) \cap A = \emptyset$, hence $\mu(B \backslash A) = 0$. We have $\mu(B) = \mu(B \cap A) + \mu(B \backslash A) + \lambda \mu(B \cap A) \mu(B \backslash A) = \mu(B \cap A)$.

3. If $\mu(T \backslash A) = 0$, we have (with 1.): $\mu(A \cup (T \backslash A)) = \mu(A)$, *i.e.* $\mu(A) = \mu(T)$, false. If $\mu(T \backslash A) = \mu(T)$, we have, for any $B \in \mathcal{C}$ (with 2.): $\mu(B) = \mu(B \cap (T \backslash A))$. For $B = A : \mu(A) = \mu(\emptyset) = 0$, false. □

Theorem 3.2. *Let $\mathcal{C} \subset \mathcal{P}(T)$ be an algebra and $\mu : \mathcal{C} \to \mathbb{R}_+$ a monotone measure with $\mu(T) = = a > 0$. The following assertions are equivalent:*

1. *μ is a λ – measure for any $\lambda \in \left(-\frac{1}{a}, \infty\right)$.*

2. *There exist* $\lambda \neq \lambda'$ *in* $\left(-\frac{1}{a}, \infty\right)$ *such that* μ *is at the same time a* λ *– measure and a* λ' *– measure.*

3. μ *is* σ *– additive and* T *is an atom of* μ*, i.e., for any* $A \in \mathcal{C}$*, one has either* $\mu(A) = 0$ *or* $\mu(A) = \mu(T) = a$.

Proof.

1. $\Rightarrow$ **2.** is obvious.

2. $\Rightarrow$ **3.** We prove first that T is an atom. Assume the contrary, *i.e.* find $A \in \mathcal{C}$ such that $0 < \mu(A) < \mu(T)$, hence $0 < \mu(T \backslash A) < \mu(T)$, according to Lemma 3.1., 3. Then $a = \mu(A) + \mu(T \backslash A) + \lambda \mu(A) \mu(T \backslash A) = \mu(A) + \mu(T \backslash A) + \lambda' \mu(A) \mu(T \backslash A) \Rightarrow 0 = (\lambda - \lambda') \mu(A) \mu(T \backslash A) \Rightarrow \lambda = \lambda'$, false.

Now, we prove that μ is σ – additive. Because $\lambda \neq \lambda'$, we can assume that $\lambda \neq 0$. Let $(E_n)_n \subset \mathcal{C}$ be a disjoint sequence such that $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$. Then

$$\mu(E) = \frac{1}{\lambda}\left(\prod_{n=1}^{\infty}(1 + \lambda \mu(E_n)) - 1\right). \quad (*)$$

In case $\mu(E_n) = 0$ for any n, one has (from $(*)$): $\mu(E) = 0 = \sum_{n=1}^{\infty} \mu(E_n)$.

In case there exists n_0 such that $\mu(E_{n_0}) > 0$ we have $\mu(E_{n_0}) = \mu(T)$. Hence $\mu(E_n) = 0$ for all $n \neq n_0$, according to Lemma 3.1, 2. Hence, from $(*)$, we get

$$\mu(E) = \frac{1}{\lambda}(1 + \lambda \mu(E_{n_0}) - 1) = \mu(T) = \sum_{n=1}^{\infty} \mu(E_n).$$

3. $\Rightarrow$ **1.** Let $\lambda \in \left(-\frac{1}{a}, \infty\right)$ arbitrarily taken. In case $\lambda = 0$, to say that μ is a 0 – measure means to say that μ is σ– additive, true.

Now, let $\lambda \neq 0$. Let $(E_n)_n \subset \mathcal{C}$ be a disjoint sequence such that $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$. In case $\mu(E_n) = 0$ for any n, equality $(*)$ is verified, because $\mu(E) = 0 = \sum_{n=1}^{\infty} \mu(E_n)$. In case there exists n_0 such that $\mu(E_{n_0}) > 0$ we have $\mu(E_{n_0}) = \mu(T)$. Hence $\mu(E_n) = 0$ for any $n \neq n_0$, because, for such n : $\mu(E_n) = \mu(E_{n_0} \cup E_n) - \mu(E_{n_0}) \leq \mu(T) - \mu(E_{n_0}) = 0$. Hence $\mu(E) = \mu(E_{n_0}) = \mu(T)$ and $(*)$ is again verified. □

Recall that, if $x \in X^{\infty}$, the Dirac measure $\delta_x : \mathcal{B} \to \mathbb{R}_+$ is defined via $\delta_x(A) = 1$, if $x \in A$ and $\delta_x(A) = 0$, if $x \notin A$. Then δ_x is σ– additive and X^{∞} is an atom of δ_x. Let us denote by *DIR* the set of all such Dirac measures, *i.e.* *DIR*$= \{\delta_x | \, x \in X^{\infty}\}$.

Theorem 3.3.
One has the equality $\left(\text{write here } \mathcal{S}_0 \overset{def}{=} \mathcal{M}\right)$

$$\bigcap_{\lambda \in (-1, \infty)} \mathcal{S}_\lambda = DIR.$$

Proof. Theorem 3.2. says that

$$\bigcap_{\lambda \in (-1, \infty)} \mathcal{S}_\lambda = \mathcal{A}$$

where $\mathcal{A}$ is the set of all $\sigma-$ additive measures $\mu : \mathcal{C} \to \mathbb{R}_+$ such that $\mu(T) = 1$ and T is an atom of μ. Because every Dirac measure is such a measure, we have the inclusion $DIR \subset \mathcal{A}$.

To prove the converse inclusion, let $m \in \mathcal{A}$ and let us find $x \in X^\infty$ such that $m = \delta_x$. We have successively:

$$1 = m(X^\infty) = \sum_{u \in X} m(uX^\infty)$$

and we find $u_1 \in X$ such that $m(u_1 X^\infty) > 0$, hence $m(u_1 X^\infty) = 1$ and $m(uX^\infty) = 0$ for $u \neq u_1$. Continuing:

$$1 = m(u_1 X^\infty) = \sum_{u \in X} m(u_1 u X^\infty)$$

and we find $u_2 \in X$ with $m(u_1 u_2 X^\infty) = 1$ and $m(uvX^\infty) = 0$ for any $uv \neq u_1 u_2$. Continuing the procedure, we find a sequence $x = u_1 u_2 ... u_n ... \in X^\infty$ such that $m(u_1 u_2 ... u_n X^\infty) = 1$ for any $n \in \mathbb{N}$.

Clearly, $\{x\} = \bigcap\limits_{n=1}^{\infty} u_1 u_2 ... u_n X^\infty$, hence $m(\{x\}) = \lim\limits_n m(u_1 u_2 ... u_n X^\infty) = 1$. Take $A \in \mathcal{B}$. If $x \in A$, one has $1 \geq m(A) \geq m(\{x\}) = 1$, hence $m(A) = 1$. If $x \notin A$, one has $m(A) = m((A \cup \{x\}) \setminus \{x\}) = m(A \cup \{x\}) - m(\{x\}) = 1 - 1 = 0$. We used the fact that $m \in \mathcal{S}_0$. Conclusion: $m = \delta_x$. □

B. We present the announced correspondence. The first basic definition is:

Definition 3.4.
Let $\lambda \in (-1, \infty) \setminus \{0\}$. A $\lambda -$ *distribution* is a sequence $(D_\lambda(n))_n$ where:

$$\begin{aligned}
D_\lambda(1) &= (a_\lambda(1), a_\lambda(2), ..., a_\lambda(p)) = (a_\lambda(i))_{1 \leq i \leq p} \\
D_\lambda(2) &= (a_\lambda(i, j))_{1 \leq i \leq p,\ 1 \leq j \leq p} \\
- - - \quad &- \quad - - - - - - - - - \\
D_\lambda(n) &= (a_\lambda(i_1, i_2, ..., i_n))_{1 \leq i_k \leq p} \quad \text{for } k = 1, 2, ..., n
\end{aligned}$$

with the following properties:

a) For any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$ one has:

- if $-1 < \lambda < 0$, then $0 < a_\lambda (i_1, i_2, ..., i_n) \le 1$;
- if $\lambda > 0$, then $a_\lambda (i_1, i_2, ..., i_n) \ge 1$.

b) $\prod_{i=1}^{p} a_\lambda (i) = \lambda + 1$

c) For any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$ one has

$$a_\lambda (i_1, i_2, ..., i_n) = \prod_{i=1}^{p} a_\lambda (i_1, i_2, ..., i_n, i).$$

Notation:
For any $\lambda \in (-1, \infty) \setminus \{0\}$, we shall denote by $\mathcal{D}_\lambda$ the set of λ – distributions.

The first correspondence is given by:

Theorem 3.5.
Let $\lambda \in (-1, \infty) \setminus \{0\}$. We have the bijection $T_\lambda : \mathcal{D}_\lambda \to \mathcal{S}_\lambda$, described as follows:

a) *Let $(D_\lambda (n))_n \in \mathcal{D}_\lambda$, where $D_\lambda (n) = (a_\lambda (i_1, i_2, ..., i_n))_{1 \le i_k \le p}$ as previously. Then $T_\lambda ((D_\lambda (n))_n) = m$, where $m (x_{i_1} x_{i_2} ... x_{i_n} X^\infty) = \dfrac{a_\lambda (i_1, i_2, ..., i_n) - 1}{\lambda}$.*

b) *The inverse $Z_\lambda = T_\lambda^{-1} : \mathcal{S}_\lambda \to \mathcal{D}_\lambda$ acts via $Z_\lambda (m) = (D_\lambda (n))_n$ where $D_\lambda (n) = (a_\lambda (i_1, i_2, ..., i_n))_{1 \le i_k \le p}$ is such that $a_\lambda (i_1, i_2, ..., i_n) = 1 + \lambda m (x_{i_1} x_{i_2} ... x_{i_n} X^\infty)$.*

Proof.

1. First, we define a bijection $L_\lambda : \mathcal{D}_\lambda \to \mathcal{D}_0$.
To this end, let $(D_\lambda (n))_n \in D_\lambda$ with $D_\lambda (n) = (a_\lambda (i_1, i_2, ..., i_n))_{1 \le i_k \le p}$. Define

$$a (i_1, i_2, ..., i_n) = \log_{\lambda+1} a_\lambda (i_1, i_2, ..., i_n).$$

We will show that, writing $(D (n))_n$ with $D (n) = (a (i_1, i_2, ..., i_n))_{1 \le i_k \le p}$, one has $(D (n))_n \in \mathcal{D}_0$, hence we have the function $L_\lambda : \mathcal{D}_\lambda \to \mathcal{D}_0$ given via

$$L_\lambda ((D_\lambda (n))_n) = (D (n))_n.$$

In any case,

$$(\lambda + 1)^{a(i_1, i_2, ..., i_n)} = a_\lambda (i_1, i_2, ..., i_n),$$

hence:

- if $-1 < \lambda < 0$, $0 < \lambda + 1 < 1$, $\log_{\lambda+1}$ is decreasing and $0 < a_\lambda (i_1, i_2, ..., i_n) \le 1$ $\Rightarrow$ $a (i_1, i_2, ..., i_n) \ge \log_{\lambda+1} 1 = 0$.
- if $\lambda > 0$, $\lambda + 1 > 1$, $\log_{\lambda+1}$ is increasing and $a_\lambda (i_1, i_2, ..., i_n) \ge 1 \Rightarrow$ $a (i_1, i_2, ..., i_n) \ge \log_{\lambda+1} 1 = 0$.

$$\sum_{i=1}^{p} a(i) = \sum_{i=1}^{p} \log_{\lambda+1} a_\lambda(i) = \log_{\lambda+1} \prod_{i=1}^{p} a_\lambda(i) = \log_{\lambda+1}(\lambda+1) = 1.$$

For any $n \in \mathbb{N}$ and any $(i_1, i_2, ..., i_n) \in U_p^n$ one has

$$\sum_{i=1}^{p} a(i_1, i_2, ..., i_n, i) = \sum_{i=1}^{p} \log_{\lambda+1} a_\lambda(i_1, i_2, ..., i_n, i) = \log_{\lambda+1} \prod_{i=1}^{p} a_\lambda(i_1, i_2, ..., i_n, i) =$$

$$= \log_{\lambda+1} a_\lambda(i_1, i_2, ..., i_n) = a(i_1, i_2, ..., i_n).$$

The next step is to show that the just defined function $L_\lambda : \mathcal{D}_\lambda \to \mathcal{D}_0$ is a bijection. The injectivity follows from the fact that $\log_{\lambda+1}$ is an injection. As for the surjectivity, pick arbitrarily $(D(n))_n \in \mathcal{D}_0$, with $D(n) = (a(i_1, i_2, ..., i_n))_{1 \le i_k \le p}$. Define $a_\lambda(i_1, i_2, ..., i_n) = \; = (\lambda+1)^{a(i_1, i_2, ..., i_n)}$ and prove that, writing $D_\lambda(n) = (a_\lambda(i_1, i_2, ..., i_n))_{1 \le i_k \le p}$, one has $(D_\lambda(n))_n \in \mathcal{D}_\lambda$.

One checks first that $-1 < \lambda < 0 \Rightarrow 0 < a_\lambda(i_1, i_2, ..., i_n) \le 1$; $\lambda > 0 \Rightarrow a_\lambda(i_1, i_2, ..., i_n) \ge 1$. Next, one sees that $\prod_{i=1}^{p} a_\lambda(i) = \prod_{i=1}^{p} (\lambda+1)^{a(i)} = (\lambda+1)^{\sum_{i=1}^{p} a(i)} = = (\lambda+1) \neq 1$. Finally,

$$\prod_{i=1}^{p} a_\lambda(i_1, i_2, ..., i_n, i) = (\lambda+1)^{\sum_{i=1}^{p} a(i_1, i_2, ..., i_n, i)} =$$

$$= (\lambda+1)^{a(i_1, i_2, ..., i_n)} = a_\lambda(i_1, i_2, ..., i_n).$$

Consequently, $(D_\lambda(n))_n \in \mathcal{D}_\lambda$.

Now, we see that $L_\lambda((D_\lambda(n))_n) = (D(n))_n$, because $\log_{\lambda+1} a_\lambda(i_1, i_2, ..., i_n) = = a(i_1, i_2, ..., i_n)$ and the surjectivity of L_λ is proved.

2. Point **b)** in the enunciation is obvious. Let us prove point **a)**. Having the bijection L_λ, we consider the schema (containing only bijections) for $\lambda \in (-1, \infty) \setminus \{0\}$:

$$\mathcal{D}_\lambda \xrightarrow{L_\lambda} \mathcal{D}_0 \xrightarrow{T_0} \mathcal{M} \xrightarrow{V_\lambda} \mathcal{S}_\lambda$$

and define $T_\lambda = V_\lambda \circ T_0 \circ L_\lambda$. Point a) in the enunciation will be proved when we will show that the action of T_λ is that one described in the enunciation. So, let us write for $(D_\lambda(n))_n \in \mathcal{D}_\lambda$ as above, $T_\lambda((D_\lambda(n))_n) = m$.

Indeed, if $D_\lambda(n) = (a_\lambda(i_1, i_2, ..., i_n))_{1 \le i_k \le p}$, we get $L_\lambda((D_\lambda(n))_n) = (D(n))_n$ with $D(n) = (a(i_1, i_2, ..., i_n))_{1 \le i_k \le p}$, namely $a(i_1, i_2, ..., i_n) = \log_{\lambda+1} a_\lambda(i_1, i_2, ..., i_n)$. Next: $T_0((D(n))_n) = \mu$, with

$$\mu(x_{i_1} x_{i_2} ... x_{i_n} X^\infty) = a(i_1, i_2, ..., i_n) = \log_{\lambda+1} a_\lambda(i_1, i_2, ..., i_n).$$

Finally, $V_\lambda(\mu) = m$, where

$$m(x_{i_1}x_{i_2}...x_{i_n}X^\infty) = \frac{(\lambda+1)^{a(i_1,i_2,...,i_n)}-1}{\lambda} = \frac{a_\lambda(i_1,i_2,...,i_n)-1}{\lambda}. \qquad \square$$

We pass to the second stage – the 'globalization'. The second basic definition is:

Definition 3.6.
A *general distribution* is a sequence $(P(n))_n$, where

$$\begin{array}{rcl} P(1) & = & (b(1), b(2), ..., b(p)) = (b(i))_{1\le i\le p} \\ P(2) & = & (b(i,j))_{1\le i\le p,\ 1\le j\le p} \\ --- & - & --------- \\ P(n) & = & (b(i_1,i_2,...,i_n))_{1\le i_k\le p} \quad \text{for } k=1,2,...,n \end{array}$$

with the following properties:

i) $\prod_{i=1}^{p} b(i) = b \in (0,\infty)\setminus\{1\}$

ii) For any $n\in\mathbb{N}$ and any $(i_1,i_2,...,i_n)\in U_p^n$, one has

- if $0<b<1$, then $0<b(i_1,i_2,...,i_n)\le 1$;
- if $b>1$, then $b(i_1,i_2,...,i_n)\ge 1$.

iii) For any $n\in\mathbb{N}$ and any $(i_1,i_2,...,i_n)\in U_p^n$, one has

$$b(i_1,i_2,...,i_n) = \prod_{i=1}^{p} b(i_1,i_2,...,i_n,i).$$

Notation: We shall denote by $\mathcal{D}$ the set of all general distributions.

Lemma 3.7. *One has the equality*

$$\mathcal{D} = \bigcup_{\lambda\in(-1,\infty)\setminus\{0\}} \mathcal{D}_\lambda,$$

the union being disjoint.

Proof. The fact that the union is disjoint follows from the Definition 3.4.

First, pick arbitrarily $\lambda\in(-1,\infty)\setminus\{0\}$ and $(D_\lambda(n))_n\in\mathcal{D}_\lambda$ and prove $(D_\lambda(n))_n \in \mathcal{D}$. Indeed, let $D_\lambda(n) = a_\lambda(i_1,i_2,...,i_n)$ and write $b = \prod_{i=1}^{n} a_\lambda(i) = \lambda+1$, hence $b\in(0,\infty)\setminus\{1\}$. One checks the properties.

To prove the converse inclusion, pick arbitrarily $(P(n))_n\in\mathcal{D}$, with $P(n) = (b(i_1,i_2,...,i_n))_{1\le i_k\le p}$. Then $b=\prod_{i=1}^{p} b(i)\in(0,\infty)\setminus\{1\}$. Define λ via $\lambda+1=b$, hence $\lambda\in(-1,\infty)\setminus\{0\}$ and check that $(P(n))_n\in\mathcal{D}_\lambda$. $\square$

Taking advantage of the fact that the union defining $\mathcal{D}$ is disjoint, we define a function $T : \mathcal{D} \to \mathcal{S}$, which will be the second correspondence (and this correspondence is global).

Definition 3.8.
The function $T : \mathcal{D} \to \mathcal{S}$ is defined via $T\left(\left(P\left(n\right)\right)_n\right) = T_\lambda\left(\left(P\left(n\right)\right)_n\right)$ if $\left(P\left(n\right)\right)_n \in$ $\in \mathcal{D}_\lambda$.
From operational point of view, we explain the definition as follows:

a) Take $\left(P\left(n\right)\right)_n \in \mathcal{D}$ arbitrarily.

b) If $P\left(1\right) = \left(b\left(i\right)\right)_{1\leq i\leq p}$, compute $b = \prod_{i=1}^{p} b\left(i\right) \in \left(0,\infty\right) \setminus \{1\}$.

c) Obtain $\lambda \in \left(-1,\infty\right) \setminus \{0\}$ from the equation $\lambda + 1 = b \Leftrightarrow \lambda = b - 1$. It follows that $\left(P\left(n\right)\right)_n \in \mathcal{D}_\lambda$.

d) Finally define $T\left(\left(P\left(n\right)\right)_n\right) = T_\lambda\left(\left(P\left(n\right)\right)_n\right)$.

Theorem 3.9.
The function T is surjective.

Proof. Let $m \in \mathcal{S}$. Then, there exists $\lambda \in \left(-1,\infty\right) \setminus \{0\}$ such that $m \in \mathcal{S}_\lambda$. Using the bijective map $T_\lambda : \mathcal{D}_\lambda \to \mathcal{S}_\lambda$, we get an (unique) $\left(D_\lambda\left(n\right)\right)_n \in \mathcal{D}_\lambda$ such that $T_\lambda\left(\left(D_\lambda\left(n\right)\right)_n\right) = T\left(\left(D_\lambda\left(n\right)\right)_n\right)$. □

The function T is not bijective, because it is not injective. We shall use the introductory results, **A.**, to study the non injectivity.

Theorem 3.10.
The function T is not injective. More precisely, for any $x \in X^\infty$, the set $T^{-1}\left(\{\delta_x\}\right)$ is infinite.

Proof. Let $x \in X^\infty$. As we have seen, $\delta_x \in \bigcap_{\lambda\in(-1,\infty)} \mathcal{S}_\lambda$, according to Theorem 3.3. For a fixed $\lambda \in \left(-1,\infty\right) \setminus \{0\}$, we can use Theorem 3.5: one has $\delta_x \in \mathcal{S}_\lambda$, hence there exists an unique $\left(D_\lambda\left(n\right)\right)_n \in \mathcal{D}_\lambda$ such that $T_\lambda\left(\left(D_\lambda\left(n\right)\right)_n\right) = \delta_x = T\left(\left(D_\lambda\left(n\right)\right)_n\right)$. And this shows that $\left(D_\lambda\left(n\right)\right)_n \in T^{-1}\left(\delta_x\right)$.

We shall construct this $\left(D_\lambda\left(n\right)\right)_n$. Write $x = x_{i_1}x_{i_2}...x_{i_n}...$ For any $n \in \mathbb{N}$, we construct $D_\lambda\left(n\right) = \left(a_\lambda\left(u_1,u_2,...,u_n\right)\right)_{1\leq u_k\leq p}$ for $k = 1,2,...,n$ as follows:

$$a_\lambda\left(u_1,u_2,...,u_n\right) = \begin{cases} \lambda+1, & \text{if } \left(u_1,u_2,...,u_n\right) = \left(i_1,i_2,...,i_n\right) \\ 1, & \text{if } \left(u_1,u_2,...,u_n\right) \neq \left(i_1,i_2,...,i_n\right) \end{cases}.$$

One can check immediately that $\left(D_\lambda\left(n\right)\right)_n \in \mathcal{D}_\lambda$.

Then, if $m = T_\lambda\left(\left(D_\lambda(n)\right)_n\right) = T\left(\left(D_\lambda(n)\right)_n\right)$ one has

$$m\left(x_{u_1}x_{u_2}...x_{u_n}X^\infty\right) = \begin{cases} \dfrac{\lambda+1-1}{\lambda} = 1, \text{ if } (u_1, u_2, ..., u_n) = (i_1, i_2, ..., i_n) \\ \dfrac{1-1}{\lambda} = 0, \qquad \text{if } (u_1, u_2, ..., u_n) \neq (i_1, i_2, ..., i_n) \end{cases}$$

This shows that $m(A) = \delta_x(A)$ for any $A \in \mathcal{P}$, *i.e.* $m = \delta_x$. We have proved that $\left(D_\lambda(n)\right)_n \in T^{-1}\left(\{\delta_x\}\right)$.

Doing this construction for all $\lambda \in (-1, \infty) \setminus \{0\}$ we get the respective $\left(D_\lambda(n)\right)_n$. Hence, one can see that $T^{-1}\left(\{\delta_x\}\right)$ is equal to the set of all $\left(D_\lambda(n)\right)_n$ constructed as above, which are different for different λ. Hence $T^{-1}\left(\{\delta_x\}\right)$ is infinite. □

4. Conclusion and future work

The results obtained in the present paper can be viewed as an instrument of work in various directions. For instance, one can perform concrete computation of Choquet and Sugeno integrals of positive measurable function (here they are sequences of positive numbers) with respect to Sugeno measures on $\mathcal{B}$. Another direction of work can be the study of cellular automata, the states of an elementary cellular automaton being either elements of X^n or of X^∞, where $X = \{0, 1\}$.

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