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An extension of ELECTRE to multi-criteria decision making problems with extended hesitant fuzzy sets

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Abstract. Extended hesitant fuzzy sets (EHFSs), which are an extension of hesitant fuzzy sets, consider possible value-groups for the membership of x into the set A and increase the richness of numerical representation based on the value-groups. Extended hesitant fuzzy set is a very useful tool in handling decision making problems, due to it can identify different decision makers (DMs) in decision making. Some extended hesitant fuzzy elements (EHFEs) aggregation operators are proposed. In this paper, the dominance relation and opposition relation for EHFEs which based on traditional ELECTRE methods are introduced and the method for comparing EHFEs is proposed. To extend EHFSs to multi-criteria decision making, we proposed an outranking approach which is similar to ELECTRE III, and the data of outranking approach is in the form of EHFEs. Then, we combine the propose method with the proposed outranking approach to ranking alternatives in multi-criteria decision making. Finally, an example is given to illustrate the developed approach.

Key-words: Extended hesitant fuzzy set; extended hesitant fuzzy elements; ELECTRE III; outranking approach; multi-criteria decision making.

1. Introduction

Since fuzzy set (FS) theory was first proposed by Zadeh[24], it has been widely studied, developed and successfully applied in various fields, such as multi-criteria decision making (MCDM)[3, 22], fuzzy logic, approximate reasoning[25] and pattern recognition[13]. Then researchers developed some extensions of fuzzy sets, such as intuitionistic fuzzy sets (IFSs)[2], type-2 fuzzy sets (T2FSs)[11], fuzzy multisets (FMSs)[23], interval-valued fuzzy sets (IVFSs)[26], interval-valued intuitionistic fuzzy sets (IVIFSs)[1], hesitant fuzzy sets (HFSs)[16], dual hesitant fuzzy sets (DHFSs)[30] and extended hesitant fuzzy sets (EHFSs)[4].

Atanassovs' IFS[2] adds an extra degree to the fuzzy sets in order to modeling the hesitant and uncertainty about the degree of membership. Type-2 fuzzy set allows the membership of a given element as a fuzzy set. In fuzzy multiset, the elements can be repeated more than once. The hesitant fuzzy set allows the membership degree having a set of possible values that make to hesitate about which one would be the right one. Extended hesitant fuzzy set consider possible value-groups for the membership. A lot of work has been done about the IFSs, FMSs, IVFSs, IVIFSs and HFSs, but, little has been done about the extended hesitant fuzzy set. Zhu[29] discussed that the hesitant fuzzy set is a particular case of the extended hesitant fuzzy set, and showed that the envelope of extended hesitant fuzzy is equivalent to the envelope of hesitant fuzzy set and envelope of an EHFE include several intuitionistic fuzzy numbers. EHFSs increase the richness of numerical representation based on the value-groups, enhance the modeling abilities of HFSs, and can identify different DMs in decision making, which expand the applications of HFSs in practice. IFSs, IVIFSs and HFSs were applied in many decision making problems[4, 5, 6, 7, 9, 10, 12, 18, 27, 28]. However, the application in multi-criteria decision making of hesitant fuzzy set, if the two DMs both assign the value 0.6, we can only save one value by the HFE, and loss the other one, which leads to information loss. Further, since generally the DMs have different importance in group decision making[20] due to their different social importance, position in the group, previous merits etc., a leading DM in a group for example, the loss of information provided by the leading DM may lead to ineffective results. EHFSs, another extension of traditional FSs, provide useful reference for our study of such situations. EHFSs were first introduced by Zhu[29].

Xia and Xu[21] defined the score function to compare HFEs, Farhadinia[7] defined a new score function to compare HFEs, which overcame the counterintuitive problem occurred in the method of Xia and Xu[21]. However, the new score function was defined on the assumption that the values in HFEs are arranged in an ascending order and if two HFEs do not have the same length, then the shorter one should be extended by adding the greatest number in it until both of the HFEs have the same length. Therefore, it is invalid in some situations which many aggregation operators have been applied in MCDM under hesitant fuzzy environment. To avoid such an issue, Wang et al.[19]proposed an outranking approach with HFSs to solve MCDM problem. The method was called relation model. It performed decision-making without a fusion method, but rather adopt outranking relations or priority functions to optimize, rank and classify alternatives in terms of priority among criteria. ELECTRE methods and PROMETHREE methods are the typical ones with in that category. ELECTRE methods[15], as representatives of relation models, are successfully and widely used in various fields due to their practical applicability. Vahdani and Hadipour[17] presented a novel ELECTRE method to solve problems that have interval-valued fuzzy information. Wang et al.[19] developed an outranking method to solve MCDM problems with hesitant fuzzy linguistic term sets. Peng et al.[14] developed an extension of ELECTRE to multi-criteria decision-making problems with multi-hesitant fuzzy sets. The research performed in this paper focuses on data characterized by a high degree of uncertainty as an extension of ELECTRE III, where these uncertainties are expressed using EHFSs. The proposed approach give a new comparative method for EHFEs based on ELECTRE methods. Furthermore, the proposed approach takes decision-makers preferences into consideration. These are realized by choosing the appropriate thresholds of the given criteria.

The remainder of the paper is organized as follows: In Section 2, we review some basic notions of hesitant fuzzy sets. In Section 3, we review extended hesitant fuzzy sets. In Section 4, a new comparative method for EHFEs are proposed and some valuable properties are also analyzed. In Section 5, an outranking approach for MCDM problems with EHFSs is presented. In Section 6, an illustrative example is provided to demonstrate the validity and feasibility of the proposed approach. Section 7 is a conclusion of the paper.

2. Preliminaries

In this section, we carry out a brief introduction to hesitant fuzzy set for a better understanding of the main body of the paper. Some operations and a comparison method for HFSs are presented.

Definition 2.1. Let X be a fixed set, a HFS on X is in terms of a function that when applied to X returns a subset of [0,1], which can be represented as the following mathematical symbol: $E = \{< x, h_E(x) >| x \in X\}$, where $h_E(x)$ is a finite set of some values in [0,1], denoting the possible membership degree of the element $x \in X$ to the set E. For convenience, the variable $h_E(x)$ is called a hesitant fuzzy element (HFE).

Example 2.1. Let $X = \{x_1, x_2, x_3\}$ is the discourse set, $A = \{< x_1, \{0.2, 0.5, 0.4\} >, < x_2, \{0.5, 0.3\} >, < x_3, \{0.2, 0.3\} >\}$, and $h = \{0.2, 0.5, 0.4\}$. Then A is a HFS on X and h is an HFE .

Given three HFEs represented by h, h_1, h_2, Torra [16] defined operations on them, which are described as :

1) $h^c = \cup_{\gamma \in h}\{1 - \gamma\}$;
2) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 \vee \gamma_2\}$;
3) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 \wedge \gamma_2\}$.

Xia and Xu[21] developed some new operations as below:

(1) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}$;
(2)$\lambda h = \cup_{\gamma \in h}\{1 - (1 - \gamma)^\lambda\}$;
(3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

Definition 2.2. [21] For an HFE h, $s(h) = \frac{1}{l_h}\sum_{\gamma \in h}\gamma$ is called the score function of h, where l_h is the number of elements in h.
For two HFEs h_1 and h_2,
if $s(h_1) > s(h_2)$, then $h_1 > h_2$;
if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

The disadvantage of using Definition 2.2 when comparing two HFEs is illustrated in the following example.

Example 2.2. Let $h_1 = \{0.4\}, h_2 = \{0.2, 0.4, 0.6\}$, and $h_3 = \{0.1, 0.3, 0.4, 0.8\}$ be three membership units, Apparently, $h_1 \neq h_2 \neq h_3$, However, according to definition 2.2, we get $s(h_1) = s(h_1) = s(h_1) = 0.4$, then $h_1 = h_2 = h_3$, which is contradictory to our intuition.

Farhadinia[7] defined a new score function, which is described as follows.

Definition 2.3. [7] Let $h = \cup_{\gamma \in h}\{\gamma\} = \{\gamma_j | j = 1, 2, \ldots, l(h)\}$ be an HFE, where $l(h)$ is the number of elements in h. Then, the score function of h is defined as

$$s(h) = \frac{\sum_{j=1}^{l(h)} \delta(j)\gamma_j}{\sum_{j=1}^{l(h)} \delta(j)}, \tag{1}$$

where $\{\delta(j) | j = 1, 2, \ldots, l(h)\}$ is a positive-valued monotonic increasing sequence of index j.

Example 2.3. Based on Example 2.2 and the new score function given by Definition 2.3, $s(h_1) = 0.4, s(h_2) = 0.52$ where h_2 becomes $\{0.2, 0.4, 0.6, 0.6\}$ as required, and $s(h_3) = 0.57$. Then $s(h_1) < s(h_2) < s(h_3)$ can be obtained and $h_1 < h_2 < h_3$.

The new score function in Definition 2.3 overcome the counterintuitive results presented in Example 2.2, however, the new score function is defined on the assumption that the values in HFEs are arranged in an ascending order and if two HFEs do not have the same length, then the shorter one should be extended by adding the greatest number in it until both of the HFEs have the same length.

3. Extended hesitant fuzzy sets

3.1. Extended hesitant fuzzy sets and Extended hesitant fuzzy elements

Definition 3.1. [29] Let X be a fixed set, $h_D(x) = \cup_{\gamma_D \in h_D(x)}\{\gamma_D\}(D = 1, \ldots, m)$ be HFEs on X. Then, an EHFS, that is H_{h_D}, is defined as: $H_{h_D}(x) = h_1(x) \times h_2(x) \times \ldots \times h_m(x) = \cup_{\gamma_1 \in h_1(x), \ldots, \gamma_m \in h_m(x)}\{< x, (\gamma_1(x), \ldots, \gamma_m(x)) > | x \in X\}$.

For convenience, we call: $H = h_1 \times \ldots \times h_m = \cup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m}\{(\gamma_1, \ldots, \gamma_m)\}$, an extended hesitant fuzzy element (EFHE). For $H = \cup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m}\{(\gamma_1, \ldots, \gamma_m)\}$, let $u = (\gamma_1, \ldots, \gamma_m)$, then we call u a membership unit (MU). Based on u, an EHFE H, can also be indicated by: [$H = \cup_{u \in H}\{u\} = \cup_{(\gamma_1, \ldots, \gamma)}\{(\gamma_1, \ldots, \gamma)\}$.

Definition 3.2. [29] Given an EHFE $H = \cup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m}\{(\gamma_1, \ldots, \gamma_m)\}$, then

$$h_H = \cup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m}\{\gamma_1, \ldots, \gamma_m\} = \cup_{\gamma \in H}\{\gamma\}$$

is called a reduced EHFE.

3.2. Basic operations

Definition 3.3. [29] Let $H = \cup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m}\{(\gamma_1, \ldots, \gamma_m)\}$ be an EHFE, then we define its complement as :

$$H^c = \cup_{(\gamma_1, \ldots, \gamma_m) \in H}\{(1 - \gamma_1, \ldots, 1 - \gamma_m)\}. \tag{2}$$

Since each u can be considered as a HFE, by the operation of HFEs, Equation 3.2 can also be denoted as :

$$H^c = \cup_{u \in H}\{u^c\}. \tag{3}$$

Given an EHFE $H = \cup_{u \in H}\{u\}$, Zhu [29] defined the minimum and maximum memberships of H, the minimum and maximum memberships of u as follows:

1) The minimum membership of $H : \gamma_H^- = \min\{\gamma | \gamma \in h_H\}$;
2) The maximum membership of $H : \gamma_H^+ = \max\{\gamma | \gamma \in h_H\}$;
3) The minimum membership of $u : u^- = \min\{\gamma | \gamma \in u\}$;
4) The maximum membership of $u : u^+ = \max\{\gamma | \gamma \in u\}$.

Definition 3.4. [29] Given two EHFEs, H_1 and H_2, the union and intersection are defined as :

$H_1 \cup H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1, u_2 | u_1^-, u_2^- \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}$;
$H_1 \cap H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1, u_2 | u_1^+, u_2^+ \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\}$.

Definition 3.5. [29] Given three EHFEs, $H = \cup_{u \in H}\{u\}, H_1 = \cup_{u_1 \in H_1}\{u_1\}$, $H_2 = \cup_{u_2 \in H_2}\{u_2\}, \lambda \geq 0$, we have:

1) $H^\lambda = \cup_{u \in H}\{u^\lambda\}$;
2) $\lambda H = \cup_{u \in H}\{\lambda u\}$;
3) $H_1 \oplus H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \oplus u_2\}$;
4) $H_1 \otimes H_2 = \cup_{u_1 \in H_1, u_2 \in H_2}\{u_1 \otimes u_2\}$.

Definition 3.6. Let $E = \{H_1, \ldots, H_n\}$ be a set of n EHFEs, Θ be a function on E, $\Theta : [0,1]^N \to [0,1]$, then $\Theta_E = \cup_{\gamma \in \{h_1 \times h_2 \times, \ldots, \times h_n\}}\{\Theta(\gamma)\}$.

Based on the operators of HFEs and the defined operations of EHFEs, we will give a series of common aggregation operator for EHFEs.

Definition 3.7. Let $H_i (i = 1, 2, \cdots, n)$ be a collection of EHFEs, an extended hesitant fuzzy weighted averaging (EHFWA)operator is a mapping $H^n \to H$ such that

EHFWA $(H_1, H_2, \ldots, H_n) = \oplus_{j=1}^n (w_j H_j) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n}\{w_1 u_1 \oplus w_2 u_2 \oplus \ldots \oplus w_n u_n\}$, where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $H_j (j = 1, 2, \ldots, n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$. Especially if $w = (\frac{1}{n}, \ldots, \frac{1}{n})$, then the EHFWA operator reduces to the extended hesitant fuzzy averaging operator :

EHFA $(H_1, H_2, \ldots, H_n) = \oplus_{j=1}^n (\frac{1}{n} H_j) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n}\{\frac{1}{n} u_1 \oplus \frac{1}{n} u_2 \oplus \cdots \oplus \frac{1}{n} u_n\}$.

Definition 3.8. Let $H_i (i = 1, 2, \cdots, n)$ be a collection of EHFEs, an extended hesitant fuzzy weighted geometric (EHFWG) operator is a mapping $H^n \to H$ such that

EHFWG$(H_1, H_2, \ldots, H_n) = \otimes_{j=1}^n (H_j^{w_j}) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n}\{u_1^{w_1} \otimes u_2^{w_2} \otimes \ldots \otimes u_n^{w_n}\}$, where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $H_j (j = 1, 2, \ldots, n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$. Especially, if $w = (\frac{1}{n}, \ldots, \frac{1}{n})$, then the EHFWG operator reduces to the extended hesitant fuzzy geometric operator :

EHFG $(H_1, H_2, \ldots, H_n) = \otimes_{j=1}^n (H_j^{w_j}) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n}\{u_1^{\frac{1}{n}} \otimes u_2^{\frac{1}{n}} \otimes \cdots \otimes u_n^{\frac{1}{n}}\}$.

Since each u can be considered as a HFE, according to Xia and Xu [21] defined operations on HFEs, the EHFWA operator and EHFWG operator can also be indicated by:

EHFWA $(H_1, H_2, \ldots, H_n) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n}\{\cup_{\gamma_i \in u_i} \prod_{j=1}^m (1 - \gamma_i)^{w_i}\}$;

EHFWG $(H_1, H_2, \ldots, H_n) = \cup_{u_1 \in H_1, \ldots, u_n \in H_n} \{\cup_{\gamma_i \in u_i} (\prod_{j=1}^{m} \gamma_i^{w_i})\}$.

Example 3.1. let $H_1 = \{(0.3, 0.4), (0.5, 0.4)\}, H_2 = \{(0.4, 0.5), (0.4, 0.7)\}$ be two EHFES, $w = (0.6, 0.4)^T$ be the weight vector of them, then by definitions, we have

$$\begin{aligned}
\text{EHFWA}\,(H_1, H_2) = \oplus_{i=1}^{2}\{w_i u_i\} &= \cup_{u_i \in H_i}\{\cup_{\gamma_{ij} \in u_i}(1 - (1 - \prod_{j=1}^{m} \gamma_{ij})^{w_i}\} \\
&= \{(1-(1-0.3)^{0.6}(1-0.4)^{0.4}, 1-(1-0.3)^{0.6}(1-0.5)^{0.4}, \\
&\quad 1-(1-0.4)^{0.6}(1-0.4)^{0.4}, 1-(1-0.4)^{0.6}(1-0.5)^{0.4}), \\
&\quad (1-(1-0.3)^{0.6}(1-0.4)^{0.4}, 1-(1-0.3)^{0.6}(1-0.7)^{0.4}, \\
&\quad 1-(1-0.4)^{0.6}(1-0.4)^{0.4}, 1-(1-0.4)^{0.6}(1-0.7)^{0.4}), \\
&\quad (1-(1-0.5)^{0.6}(1-0.4)^{0.4}, 1-(1-0.5)^{0.6}(1-0.5)^{0.4}, \\
&\quad 1-(1-0.4)^{0.6}(1-0.4)^{0.4}, 1-(1-0.4)^{0.6}(1-0.4)^{0.5}), \\
&\quad (1-(1-0.5)^{0.6}(1-0.4)^{0.4}, 1-(1-0.5)^{0.6}(1-0.7)^{0.4}, \\
&\quad 1-(1-0.4)^{0.6}(1-0.4)^{0.4}, 1-(1-0.4)^{0.6}(1-0.7)^{0.4})\} \\
&= \{(0.3419, 0.3881, 0.400, 0.4422), (0.3419, 0.5012, 0.400, 0.5453), \\
&\quad (0.4622, 0.500, 0.400, 0.4422), (0.4622, 0.5924, 0.400, 0.5453)\}.
\end{aligned}$$

$$\begin{aligned}
\text{EHFWG}\,(H_1, H_2) = \otimes_{i=1}^{2}\{u_i^{w_i}\} &= \cup_{u_i \in H_i}\{\cup_{\gamma_{ij} \in u_i}(\prod_{j=1}^{m} \gamma_{ij}^{w_i})\} \\
&= \{(0.3^{0.6} \times 0.4^{0.4}, 0.3^{0.6} \times 0.5^{0.4}, 0.4^{0.6} \times 0.4^{0.4}, 0.4^{0.6} \times 0.5^{0.4}), \\
&\quad (0.3^{0.6} \times 0.4^{0.4}, 0.3^{0.6} \times 0.7^{0.4}, 0.4^{0.6} \times 0.4^{0.4}, 0.4^{0.6} \times 0.7^{0.4}), \\
&\quad (0.5^{0.6} \times 0.4^{0.4}, 0.5^{0.6} \times 0.5^{0.4}, 0.4^{0.6} \times 0.4^{0.4}, 0.4^{0.6} \times 0.5^{0.4}), \\
&\quad (0.5^{0.6} \times 0.4^{0.4}, 0.5^{0.6} \times 0.7^{0.4}, 0.4^{0.6} \times 0.4^{0.4}, 0.5^{0.6} \times 0.7^{0.4})\} \\
&= \{(0.3366, 0.3680, 0.4000, 0.4373), (0.3366, 0.3680, 0.4000, 0.5004), \\
&\quad (0.4573, 0.5000, 0.4000, 0.4373), (0.4573, 0.5720, 0.4000, 0.5004)\}.
\end{aligned}$$

4. Outranking relations on EHFEs

In ELECTRE III method, for the jth criterion being considered, the concordance index and the discordance index are constructed with three associated thresholds: the preference threshold p_j, the indifference threshold q_j, and the veto threshold v_j. Among those three thresholds, p_j is used to justify the preference in favor of either of the actions, q_j stands for being compatible with indifference between two actions, and v_j is assigned to introduce discordance into the outranking relations. Note that, in this paper, we only consider the simple case where the thresholds p_j, q_j and v_j are constants under each criterion. This simplification of using constant thresholds aids the illustration of ELECTRE III method and our approach. Actually, they can be generalized to functions varying with the value of the criteria $g_j(a)$, that is, the case of variable thresholds $p_j(g_j(a))$, $q_j(g_j(a))$ and $v_j(g_j(a))$[15].

Definition 4.1. [15] Let G be the criteria set $G = \{g_1, \dots, g_j, \dots, g_m\}$, B be the set of alternatives or actions $B = \{a_1, \dots, a_i, \dots, a_n\}$. Two thresholds under the criterion g_j have been specified to construct the fuzzy concordance index: the indifference threshold q_j and the preference threshold $p_j (0 \leq q_j < p_j)$. Let a_1 and a_2 be two alternatives, where $a_1, a_2 \in B$, and then the relations can be defined as follows.

(1) If $g_j(a_1) - g_j(a_2) \geq p_j$, then a_1 is strictly preferred to a_2, denoted by $P(a_1, a_2)$;
(2) If $q_j < g_j(a_1) - g_j(a_2) < p_j$, then a_1 is weakly preferred to a_2, denoted by $W(a_1, a_2)$;
(3) If $|g_j(a_1) - g_j(a_2)| \leq q_j$, then a_1 is indifferent to a_2, denoted by $I(a_1, a_2)$.

The concordance index for the single criterion is defined as follows.
(1) If $g_j(a_1) + q_j \geq g_j(a_2)$, then $c_j(a_1, a_2) = 1$;
(2) If $g_j(a_1) + q_j < g_j(a_2) < g_j(a_1) + p_j$, then $c_j(a_1, a_2) = \frac{g_j(a_1) - g_j(a_2) + p_j}{p_j - q_j}$;
(3) If $g_j(a_1) + p_j \leq g_j(a_2)$, then $c_j(a_1, a_2) = 0$.

Definition 4.2. [15] A veto threshold $v_j (> p_j)$ is introduced based on Definition 4.1. Then, the discordance index $d(a_1, a_2)$ is defined as follows.
(1) If $g_j(a_2) - g_j(a_1) \leq p_j$, then $d_j(a_1, a_2) = 0$;
(2) If $g_j(a_1) + p_j < g_j(a_2) < g_j(a_1) + v_j$, then $c_j(a_1, a_2) = \frac{g_j(a_2) - g_j(a_1) - p_j}{v_j - p_j}$;
(3) If $g_j(a_1) + v_j \geq g_j(a_2)$, then $d_j(a_1, a_2) = 1$.

Following the rules of ELECTRE III method given by the outranking relation, Peng [14] defined concordance index and discordance index for multi-hesitant fuzzy elements.

Definition 4.3. [19] Let h_1, h_2 be two multi-hesitant fuzzy elements, $p, q (q < p)$ be two thresholds, the concordance index can be defined as follows:
$r_{p,q}(h_1, h_2) = \frac{1}{l(h_1)} \sum_{\gamma_1 \in h_1} \min_{\gamma_2 \in h_2} \{c_{p,q}(\gamma_1, \gamma_2)\}$.
Where $l(h_1)$ is the number of elements in h_1 and $c_{p,q}(\gamma_1, \gamma_2)$ is the concordance index for the values γ_1 and γ_2 under thresholds p and q.

Definition 4.4. [19] Let h_1, h_2 be two multi-hesitant fuzzy elements, $p, v (p < v)$ be two thresholds, the concordance index can be defined as follows:
$t_{p,v}(h_1, h_2) = \frac{1}{l(h_1)} \sum_{\gamma_1 \in h_1} \min_{\gamma_2 \in h_2} \{d_{p,v}(\gamma_1, \gamma_2)\}$.
Where $l(h_1)$ is the number of elements in h_1 and $d_{p,v}(\gamma_1, \gamma_2)$ is the concordance index for the values γ_1 and γ_2 under thresholds p and v.

Definition 4.5. Let $H_1, H_2 \in$ EHFEs, $p, q (q < p)$ be two thresholds, then we define the concordance index for EHFEs as follows :

$$r_{p,q}(H_1, H_2) = \frac{1}{l(H_1) \cdot l(H_2)} \sum_{u_1 \in H_1, u_2 \in H_2} r_{p,q}(u_1, u_2), \tag{4}$$

$$r_{p,q}(u_1, u_2) = \frac{1}{l(u_1)} \sum_{\gamma_1 \in u_1} \min_{\gamma_2 \in u_2} \{c_{p,q}(\gamma_1, \gamma_2)\}. \tag{5}$$

Here $l(H_1)$ is the number of elements in H_1, $l(H_2)$ is the number of elements in H_2, $l(u_1)$ is the number of elements in u_1 and $c_{p,q}(\gamma_1, \gamma_2)$ is the concordance index for the values γ_1 and

γ_2 under thresholds p and q. It is simple to ascertain that, if both of H_1 and H_2 have only a single value, $u_1 \in H_1$ and $u_2 \in H_2$ also have a single value, $r_{p,q}(H_1, H_2)$ will return into a concordance index, as introduce in Definition 4.1.

Property 4.1. Let H_1, H_2 be two EHFEs, p and $q(p \geq q \geq 0)$ be two thresholds, and then

$$0 \leq r_{p,q}(H_1, H_2) \leq 1. \tag{6}$$

Definition 4.6. The strict dominance relation, the weak dominance relation and the indifference relation of EHFEs can be defined as follows.

(1) If $r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) = 1$(which is equivalent to $r_{p,q}(H_1, H_2) = 1$ and $r_{p,q}(H_2, H_1) = 0$), then H_1 strongly dominates H_2 (H_2 is strongly dominated by H_1), denoted by $H_1 >_S H_2$;

(2) If $r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) = 0$, then H_1 is indifferent to H_2, denoted by $H_1 \sim H_2$;

(3) If $0 < r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) < 1$, then H_1 weakly dominates H_2 (H_2 is weakly dominated by H_1), denoted by $H_1 >_W H_2$;

(4) If $0 < r_{p,q}(H_2, H_1) - r_{p,q}(H_1, H_2) < 1$, then H_2 weakly dominates H_1 (H_1 is weakly dominated by H_2), denoted by $H_2 >_W H_1$.

Example 4.1. Let $p = 0.1, q = 0.05$.

(1) If $H_1 = \{(0.35, 0.35, 0.45), (0.35, 0.4, 0.45)\}, H_2 = \{(0.1, 0.2, 0.3), (0.1, 0.2, 0.2)\}$, then $r_{p,q}(H_1, H_2) = 1, r_{p,q}(H_2, H_1) = 0$, so $H_1 >_S H_2$;

(2) If $H_1 = \{(0.26, 0.3, 0.3), (0.26, 0.3, 0.35)\}, H_2 = \{(0.21, 0.21, 0.3), (0.21, 0.22, 0.3)\}$, then $r_{p,q}(H_1, H_2) = 1, r_{p,q}(H_2, H_1) = 1/12$, so $H_1 >_W H_2$;

(3) If $H_1 = \{(0.25, 0.28, 0.3), (0.28, 0.28, 0.3)\}, H_2 = \{(0.25, 0.26, 0.3), (0.25, 0.25, 0.3)\}$, then $r_{p,q}(H_1, H_2) = 1, r_{p,q}(H_2, H_1) = 1$, so $H_1 \sim H_2$.

Property 4.2. Let H_1, H_2 be two EHFEs, p and $q(p \geq q \geq 0)$ be two thresholds, and then $H_1 >_S H_2$ if and only if $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$.

Proof. (1) Necessity: $H_1 >_S H_2 \Rightarrow \min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$.

According to Definition 4.3, if $H_1 >_S H_2$, then $r_{p,q}(H_1, H_2) - r_{p,q}(H_2, H_1) = 1$, because $r_{p,q}(H_1, H_2)$ and $r_{p,q}(H_2, H_1)$ are in [0,1], so $r_{p,q}(H_1, H_2) = 1$ and $r_{p,q}(H_2, H_1) = 0$ can be obtained. Thus for any $u_1 \in H_1, u_2 \in H_2, r_{p,q}(u_2, u_1) = \frac{1}{l(u_2)} \sum_{\gamma_2 \in u_2} \min_{\gamma_1 \in u_1}\{c_{p,q}(\gamma_2, \gamma_1)\} = 0$ is obtained. As derived from Definition 4.1, $c_{p,q}(\gamma_2, \gamma_1) \in [0, 1]$, so for any $\gamma_1 \in u_1, \gamma_2 \in u_2$, we have $\gamma_1 - \gamma_2 \geq p$. Therefor, $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$ is certainly valid.

(2) Sufficiency: $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p \Rightarrow H_1 >_S H_2$.

Because of $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$, then $\gamma_1 - \gamma_2 \geq p$ for any $\gamma_1 \in H_1, \gamma_2 \in H_2$. According to Definition 4.1, $c_{p,q}(\gamma_1, \gamma_2) = 1, c_{p,q}(\gamma_2, \gamma_1) = 0$. Therefore, $r_{p,q}(u_1, u_2) = 1, r_{p,q}(u_2, u_1) = 0$ for any $u_1 \in H_1, u_2 \in H_2$, which indicate that $r_{p,q}(H_2, H_1) = 0, r_{p,q}(H_1, H_2) = 1$. Then, according to Definition 4.5, $H_1 >_S H_2$.

Property 4.3. Let H_1, H_2, H_3 be three EHFEs, p and $q(p \geq q \geq 0)$ be two thresholds. If $H_1 >_S H_2, H_2 >_S H_3$, then $H_1 >_S H_3$.

Proof. According to Property 4.2, if $H_1 >_S H_2, H_2 >_S H_3$, then $\min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$, $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_3|\gamma_3 \in H_3\} \geq p$, so $2p \leq \min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_2|\gamma_2 \in H_2\} + \min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_3|\gamma_3 \in H_3\} \leq \min\{\gamma_1|\gamma_1 \in H_1\} - \max\{\gamma_3|\gamma_3 \in H_3\}$. Therefore $H_1 >_S H_3$.

Definition 4.7. Let $H_1, H_2 \in$ EHFEs, $p, v(p < v)$ be two thresholds, then we define the discordance index for EHFEs as follows :

$$t_{p,v}(H_1, H_2) = \frac{1}{l(H_1) \cdot l(H_2)} \sum_{u_1 \in H_1, u_2 \in H_2} t_{p,v}(u_1, u_2), \tag{7}$$

$$t_{p,v}(u_1, u_2) = \frac{1}{l(u_1)} \sum_{\gamma_1 \in u_1} \min_{\gamma_2 \in u_2} \{d_{p,v}(\gamma_1, \gamma_2)\}. \tag{8}$$

Here $l(H_1)$ is the number of elements in H_1, $l(H_2)$ is the number of elements in H_2, $l(u_1)$ is the number of elements in u_1 and $d_{p,v}(\gamma_1, \gamma_2)$ is the discordance index for the values γ_1 and γ_2 under thresholds p and v. It is simple to ascertain that, if both of H_1 and H_2 have only a single value, $u_1 \in H_1$ and $u_2 \in H_2$ also have a single value, $t_{p,v}(H_1, H_2)$ will return into a discordance index, as introduce in Definition 4.2.

Property 4.4. Let H_1, H_2 be two EHFEs, p and $v(p < v)$ be two thresholds, and then

$$0 \leq t_{p,v}(H_1, H_2) \leq 1 \tag{9}$$

Definition 4.8. The strong opposition relation, weak opposition relation and indifferent opposition relation of EHFEs can be defined as follows.

(1) If $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 1$(which is equivalent to $t_{p,v}(H_1, H_2) = 1$ and $t_{p,v}(H_2, H_1) = 0$), then H_1 strongly opposes H_2 (H_2 is strongly opposed by H_1), denoted by $H_1 >_{SO} H_2$;

(2) If $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 0$, then H_1 is indifferent opposed to H_2, denoted by $H_1 \sim_O H_2$;

(3) If $0 < t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) < 1$, then H_1 weakly opposes H_2 (H_2 is weakly opposed by H_1), denoted by $H_1 >_{WO} H_2$;

(4) If $0 < t_{p,v}(H_2, H_1) - t_{p,v}(H_1, H_2) < 1$, then H_2 weakly opposes H_1 (H_1 is weakly opposed by H_2), denoted by $H_2 >_{WO} H_1$.

Property 4.5. Let H_1, H_2 be two EHFEs, p and $v(p < v)$ be two thresholds, and then $H_1 >_{SO} H_2$ if and only if $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p$.

Proof. (1) Necessity: $H_1 >_{SO} H_2 \Rightarrow \min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p$.

According to Definition 4.8, if $H_1 >_{SO} H_2$, then $t_{p,v}(H_1, H_2) - t_{p,v}(H_2, H_1) = 1$, because $t_{p,v}(H_1, H_2)$ and $t_{p,v}(H_2, H_1)$ are in [0,1], so $t_{p,v}(H_1, H_2) = 1$ and $t_{p,v}(H_2, H_1) = 0$ can be obtained. Thus for any $u_1 \in H_1, u_2 \in H_2, t_{p,v}(u_1, u_2) = \frac{1}{l(u_1)} \sum_{\gamma_1 \in u_1} \min_{\gamma_2 \in u_2} \{d_{p,v}(\gamma_1, \gamma_2)\} = 1$ is obtained. As derived from Definition 4.2, $d_{p,v}(\gamma_1, \gamma_2) \in [0, 1]$, so for any $\gamma_1 \in u_1, \gamma_2 \in u_2$, we have $\gamma_2 - \gamma_1 \geq p$. Therefore, $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p$ is certainly valid.

(2) Sufficiency: $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p \Rightarrow H_1 >_{SO} H_2$.

Because of $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p$, then $\gamma_2 - \gamma_1 \geq p$ for any $\gamma_1 \in H_1, \gamma_2 \in H_2$. According to Definition 4.2, $d_{p,v}(\gamma_1,\gamma_2) = 1, d_{p,v}(\gamma_2,\gamma_1) = 0$. Therefore, $t_{p,v}(u_1,u_2) = 1, t_{p,v}(u_2,u_1) = 0$ for any $u_1 \in H_1, u_2 \in H_2$, which indicate that $t_{p,v}(H_2,H_1) = 0, t_{p,v}(H_1,H_2) = 1$. Then, according to Definition 4.8, $H_1 >_{SO} H_2$.

Property 4.6. Let H_1, H_2, H_3 be three EHFEs, p and $v(p < v)$ be two thresholds. If $H_1 >_{SO} H_2, H_2 >_{SO} H_3$, then $H_1 >_{SO} H_3$.
Proof. According to Property 4.5, if $H_1 >_{SO} H_2, H_2 >_{SO} H_3$, then $\min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \geq p$, $\min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_2|\gamma_2 \in H_2\} \geq p$, so $2p \leq \min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_2|\gamma_2 \in H_2\} + \min\{\gamma_2|\gamma_2 \in H_2\} - \max\{\gamma_1|\gamma_1 \in H_1\} \leq \min\{\gamma_3|\gamma_3 \in H_3\} - \max\{\gamma_1|\gamma_1 \in H_1\}$. Therefore $H_1 >_{SO} H_3$.

5. An ELECTRE approach for MCDM problems with EHFEs

Let $A = \{A_1,\ldots,A_m\}$ be the set of alternatives, $C = \{C_1,\ldots,C_n\}$ be the set of criteria and $E = \{e_1,\ldots,e_r\}$ be the set of decision makers. Suppose that the decision maker e_k provides all the possible evaluated values under the criterion C_j for the alternative A_i denoted by a HFE $h_{ij}^{(k)}$ and we constructs the decision matrix $D^k = (h_{ij}^{(k)})_{m\times n}$ given by kth decision maker. The weight of the decision maker e_k is $\tilde{w}_k$, where $k = 1,\ldots,r, \sum_{k=1}^{r} \tilde{w}_k = 1$. The weight of the criteria C_j is w_j, where $j = 1,\ldots,m, \sum_{j=1}^{m} w_j = 1$. Our method is an integration of EHFSs and ELECTRE III to solve MCDM problems mention above, and we set the thresholds q_j, p_j and $v_j(q_j < p_j < v_j)$ associated with the criterion C_j.
Step 1. Determine the weighted hesitant fuzzy decision matrix $\tilde{D}^{(k)} = (\tilde{h}_{ij}^{(k)})_{m\times n}$.

According to the weights of the decision maker and the operations defined by Xia and Xu [25], the weighted hesitant fuzzy decision matrix can be constructed using the following formula:

$$\tilde{h}_{ij}^{(k)} = \tilde{w}_k h_{ij}^{(k)} \tag{10}$$

Step 2. Construct the extended hesitant fuzzy decision matrix $H = (H_{ij})_{m\times n}$ by EHFEs $H_{ij}(i = 1,\ldots,m; j = 1,\ldots,n)$.
Step 3. Determine the thresholds and the concordance set of subscripts.

We set the thresholds q_j, p_j and $v_j(0 \leq q_j < p_j < v_j)$ associate with the criterion C_j. For convenience of calculation, in this paper, we take $q_i = 0.05, p_i = 0.1, v_i = 0.2$.
The concordance set of subscripts, which should satisfy the constrain $H_{ij} >_S H_{sj}$ or $H_{ij} >_W H_{sj}$ or $H_{ij} \sim H_{sj}$ or $H_{sj} >_W H_{ij}$ is represented as follows:

$$O(A_i, A_s) = \{j|1 \leq j \leq n, H_{ij} >_Z H_{sj} or H_{sj} >_W H_{ij}\}(i, s = 1,\ldots,m). \tag{11}$$

Here, where $Z = \{S, W, I\}$, $H_{ij} >_S H_{sj}$ means H_{ij} strongly dominates H_{sj},$H_{ij} >_W H_{sj}$ means H_{ij} weakly dominates H_{sj},$H_{ij} >_I H_{sj}$ means H_{ij} is different to H_{sj}.
Step 4. Determine the concordance matrix.

Using the weight vector $w = (w_1,\ldots,w_n)$ associated with the criteria, the concordance index $C(A_i, A_s)$ is represented as follows:

$$C(A_i, A_s) = \sum_{j\in O(A_i,A_s)} w_j r_{p,q}(H_{ij}, H_{sj}), \tag{12}$$

$C(A_i, A_s)$ ranges from 0 to 1, a value of 0 indicates that alternative A_i is worse than alternative A_s.
Then, the concordance matrix C can be constructed as

$$C = \begin{bmatrix} - & C_{12} & \cdots & C_{1m} \\ C_{22} & - & \cdots & C_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ C_{m1} & C_{m2} & \cdots & - \end{bmatrix}. \tag{13}$$

Step 5. Determine the credibility index of outranking relations.

$$\sigma(A_i, A_s) = \begin{cases} C(A_i, A_s), & If \quad F = \emptyset \\ C(A_i, A_s)\prod_{j\in F} \frac{1-t_{p,v}(H_{ij},H_{sj})}{1-C(A_i,A_s)}, & If \quad F \neq \emptyset, \end{cases} \tag{14}$$

where $F = \{j|t_{p,v}(H_{ij}, H_{sj}) > C(A_i, A_s)\}$.
Step 6. Rank all alternatives.

The ranking index of the alternatives is defined as follows:

$$\delta(A_i) = \sum_{s=1}^{m} \sigma(A_i, A_s) - \sum_{s=1}^{m} \sigma(A_s, A_i). \tag{15}$$

Under such a correlation, the larger $\delta(A_i)$ is, the better the alternative A_i is.

6. Illustrative example

Example 6.1. This example is adapted from[8]. Energy is an indispensable factor for the social-economic development of societies. Thus the correct energy policy affects economic development and environment, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) $A_i(i = 1, 2, 3, 4, 5)$ to be invested, and four criteria to be considered: S_1: technology; S_2: environmental; S_3: socio-political; S_4: economic. Five DMs are invited to evaluate the performance of the five alternatives.

Zhu and Xu[29] used EHFSs to collect the DMs' preferences and utilized the extended hesitant distance measures to deal with this problem. In this paper we develop a new method to deal with this energy policy problem, which consider weights of the DMs in group decision making.
Step 1: The DMs $D_k(k = 1, 2, 3, 4, 5)$ provide their preference over all the alternatives $A_i(i = 1, 2, 3, 4, 5)$ with respect to the criteria $S_j(j = 1, 2, 3, 4)$ based on HFSs, then we can construct EHFSs, and get an extended hesitant fuzzy matrix $H = (H_{ij})_{5\times4}$, which indicates the group preferences over the alternatives A_i of the criteria S_j. Assume that the matrix is shown in Table 1.
Step 2. Let $\omega = (0.3, 0.1, 0.3, 0.2, 0.1)$ be the weighting vector of the DMs, then we can calculate the weighted extended hesitant fuzzy decision matrix $H = (H_{ij})_{m\times n}$ by EHFEs $H_{ij}(i = 1, \ldots, m; j = 1, \ldots, n)$ and the matrix is shown in Table 2.
Step 3. Determine the thresholds and the concordance set of subscripts.

According to Equation 4.1 and Equation 4.2, we obtain $H_{21} >_W H_{11}$ and $H_{23} >_W H_{13}$, so $O(A_1, A_2) = \{1, 3\}$. Similarly, the concordance set of subscripts can be determined as follows:

Table 1: The extended hesitant fuzzy decision making matrix

	S_1	S_2
A_1	$\{(0.3, 0.4, 0.3, 0.4, 0.5)\}$	$\{(0.7, 0.8, 0.3, 0.8, 0.6), (0.7, 0.8, 0.4, 0.8, 0.6)\}$
A_2	$\{(0.3, 0.4, 0.5, 0.2, 0.5), (0.3, 0.4, 0.5, 0.3, 0.5)\}$	$\{(0.5, 0.6, 0.5, 0.6, 0.6)\}$
A_3	$\{(0.4, 0.5, 0.5, 0.5, 0.6)\}$	$\left\{ \begin{array}{l} (0.5, 0.6, 0.7, 0.6, 0.5), (0.6, 0.6, 0.7, 0.6, 0.5), \\ (0.5, 0.6, 0.8, 0.6, 0.5), (0.6, 0.6, 0.8, 0.6, 0.5) \end{array} \right\}$
A_4	$\{(0.3, 0.2, 0.2, 0.3, 0.1)\}$	$\{(0.6, 0.5, 0.7, 0.5, 0.5)\}$
A_5	$\{(0.3, 0.4, 0.6, 0.2, 0.2), (0.3, 0.3, 0.6, 0.2, 0.2)\}$	$\{(0.6, 0.8, 0.5, 0.4, 0.6), (0.6, 0.8, 0.5, 0.5, 0.6)\}$
	S_3	S_4
A_1	$\left\{ \begin{array}{l} (0.3, 0.4, 0.2, 0.3, 0.2), (0.4, 0.4, 0.2, 0.3, 0.2), \\ (0.3, 0.4, 0.3, 0.3, 0.2), (0.4, 0.4, 0.3, 0.3, 0.2) \end{array} \right\}$	$\{(0.6, 0.5, 0.5, 0.4, 0.6)\}$
A_2	$\{(0.6, 0.4, 0.5, 0.3, 0.5), (0.6, 0.4, 0.4, 0.3, 0.5)\}$	$\{(0.3, 0.4, 0.5, 0.2, 0.2), (0.3, 0.4, 0.4, 0.2, 0.2)\}$
A_3	$\{(0.7, 0.3, 0.9, 0.8, 0.6), (0.7, 0.3, 0.8, 0.8, 0.6)\}$	$\{(0.7, 0.8, 0.7, 0.8, 0.8)\}$
A_4	$\{(0.4, 0.3, 0.2, 0.3, 0.5)\}$	$\{(0.3, 0.2, 0.7, 0.2, 0.1)\}$
A_5	$\{(0.7, 0.5, 0.6, 0.8, 0.6)\}$	$\{(0.6, 0.4, 0.5, 0.4, 0.6), (0.7, 0.4, 0.5, 0.4, 0.6)\}$

Table 2: The weighted extended hesitant fuzzy decision making matrix

	S_1	S_2
A_1	$\{(0.09, 0.04, 0.09, 0.08, 0.05)\}$	$\{(0.21, 0.08, 0.09, 0.16, 0.06), (0.21, 0.08, 0.12, 0.16, 0.06)\}$
A_2	$\{(0.09, 0.04, 0.15, 0.04, 0.05), (0.09, 0.04, 0.15, 0.06, 0.05)\}$	$\{(0.15, 0.06, 0.15, 0.12, 0.06)\}$
A_3	$\{(0.12, 0.05, 0.15, 0.1, 0.06)\}$	$\left\{\begin{array}{c}(0.15, 0.06, 0.21, 0.12, 0.05), (0.18, 0.06, 0.21, 0.12, 0.05), \\ (0.15, 0.06, 0.24, 0.12, 0.05), (0.18, 0.06, 0.24, 0.12, 0.05)\end{array}\right\}$
A_4	$\{(0.09, 0.02, 0.06, 0.06, 0.01)\}$	$\{(0.18, 0.05, 0.21, 0.1, 0.05)\}$
A_5	$\{(0.09, 0.04, 0.18, 0.04, 0.02), (0.09, 0.03, 0.18, 0.04, 0.02)\}$	$\{(0.18, 0.08, 0.15, 0.08, 0.06), (0.18, 0.08, 0.15, 0.1, 0.06)\}$
	S_3	S_4
A_1	$\left\{\begin{array}{c}(0.09, 0.04, 0.06, 0.06, 0.02), (0.12, 0.04, 0.06, 0.06, 0.02), \\ (0.09, 0.04, 0.09, 0.06, 0.02), (0.12, 0.04, 0.09, 0.06, 0.02)\end{array}\right\}$	$\{(0.18, 0.05, 0.15, 0.08, 0.06)\}$
A_2	$\{(0.18, 0.04, 0.15, 0.06, 0.05), (0.18, 0.04, 0.12, 0.06, 0.05)\}$	$\{(0.09, 0.04, 0.15, 0.04, 0.02), (0.09, 0.04, 0.12, 0.04, 0.02)\}$
A_3	$\{(0.21, 0.03, 0.27, 0.16, 0.06), (0.21, 0.03, 0.24, 0.16, 0.06)\}$	$\{(0.21, 0.08, 0.21, 0.16, 0.08)\}$
A_4	$\{(0.12, 0.03, 0.06, 0.06, 0.05)\}$	$\{(0.09, 0.02, 0.21, 0.04, 0.01)\}$
A_5	$\{(0.21, 0.05, 0.18, 0.16, 0.06)\}$	$\{(0.18, 0.04, 0.15, 0.08, 0.06), (0.21, 0.04, 0.15, 0.08, 0.06)\}$

$$O = (O(A_i, A_s)) = \begin{bmatrix} - & 1,3 & 1,3,4 & 3 & 1,3 \\ 2,4 & - & 1,2,3,4 & 4 & 1,2,3,4 \\ 2 & - & - & - & - \\ 1,2,4 & 1,2,3 & 1,2,3,4 & - & 1,3,4 \\ 2,4 & - & 1,2,3,4 & 2 & - \end{bmatrix}.$$

Step 4. Determine the concordance matrix.

We assume that the weight vector of the attributes S_j is $\omega = \{0.25, 0.25, 0.25, 0.25)^T$, and according to Equation 5.3, the concordance index $C(A_1, A_2) = \sum_{j \in O(A_1, A_2)} w_j r_{p,q}(H_{1j}, H_{2j}) = w_1 r_{p,q}(H_{11}, H_{21}) + w_3 r_{p,q}(H_{13}, H_{23}) = 0.444$.
Similarly, the concordance matrix is :

$$C = \begin{bmatrix} - & 0.444 & 0.6 & 0.228 & 0.332 \\ 0.396 & - & 0.712 & 0.204 & 0.76 \\ 0.202 & - & - & - & - \\ 0.604 & 0.544 & 0.622 & - & 0.488 \\ 0.4 & - & 0.733 & 0.192 & - \end{bmatrix}.$$

Step 5. Determine the credibility index of outranking relations.

Calculate $t_{p,v}(H_{ij}, H_{sj})(i = 1, \ldots, m; s = 1, \ldots, m)$. According to Step 4 and Equation 5.5, the credibility index matrix can be determine as follows:

$$\sigma = \begin{bmatrix} - & 0.444 & 0.6 & 0.228 & 0.332 \\ 0.396 & - & 0.712 & 0.204 & 0.76 \\ 0.194 & - & - & - & - \\ 0.604 & 0.544 & 0.622 & - & 0.488 \\ 0.4 & - & 0.733 & 0.192 & - \end{bmatrix}.$$

Step 6. Rank all alternatives.

Using Equation 5.6, we calculate the values of $\delta(A_i)(i = 1, 2, 3, 4, 5)$. They are $\delta(A_1) = -0.01, \delta(A_2) = 1.084, \delta(A_3) = -2.473, \delta(A_4) = 1.632, \delta(A_5) = -0.255$. Thus, the ranking of the alternatives are: $A_4 \succ A_2 \succ A_1 \succ A_3 \succ A_5$, and the most desirable alternative is A_4.

Compared to[29], the ranking $A_4 \succ A_2 \succ A_1 \succ A_5 \succ A_3$ is changed to $A_4 \succ A_2 \succ A_1 \succ A_3 \succ A_5$. And compared with hesitant fuzzy sets, the extended hesitant fuzzy sets avoid information loss and thus retain the preference information of each DM when dealing with multi-attribute decision-making problems. What's more, in this paper, we propose an ELECTRE approach for MCDM problems with EHFEs, which can rank all alternatives and choose the best one according to extended hesitant fuzzy information given by the DMs. And the proposed method in this paper considers both the weights of decision makers and attributes, and the final result $\delta(A_i)$ can be either positive or negative. However, in[29], The distance measure between alternatives only considers the decision maker's attributes and its result can only be on [0,1]. Therefore, the method we proposed considers information comprehensively and can be applied to many fields.

7. Conclusion

The EHFS theory is a useful to deal with MCDM problems, when there are multiple decision makers and DMs have different importance in group decision making. In this paper, we first review the theory of EHFSs, then, we proposed the domination relation and opposition relation for EHFEs motivated by the idea of traditional ELECTRE methods. An outranking approach is proposed to solve MCDM problems, the advantages of the developed approach are that we avoid the comparison of EHFEs and the elimination of shortcomings of regular operations. In this paper, the thresholds p, q and v are considered as constants to each criterion, but we only assumed that they have exact value. EHFSs plays an importance role in group decision making with more studies on the theory and applications.

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References

[1] ATANASSOV K., GARGOV G., *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol. 31, no. 3, pp. 343–349, 1989.

[2] ATANASSOV K. T., RANGASAMY P., *Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems*, Vol. 20, no. 1, pp. 87C-96, 1986.

[3] BELLMAN R. E., ZADEH L. A., *Decision-Making in a Fuzzy Environment*, Management Science, Vol. 17, no. 4, pp. 141–164, 1970.

[4] CHEN T., *A comparative analysis of score functions for multiple criteria decision making in intuitionistic fuzzy settings*, Information Sciences, Vol. 181, no. 17, pp. 3652–3676, 2011.

[5] CHEN T., *An outcome-oriented approach to multicriteria decision analysis with intuitionistic fuzzy optimistic pessimistic operators*, Expert Systems with Applications, Vol. 37, no. 12, pp. 7762–7774, 2010.

[6] CHEN J., HUANG X., TANG J., *Distance measures for higher order dual hesitant fuzzy sets*, Computational and Applied Mathematics, Vol. 37, no. 2, pp. 1784-1806, 2018.

[7] FARHADINIA B., *A Novel Method of Ranking Hesitant Fuzzy Values for Multiple Attribute Decision-Making Problems*, International Journal of Intelligent Systems, Vol. 28, no. 8, pp. 752–767, 2013.

[8] KAHRAMAN C., IHSAN K., *A fuzzy multicriteria methodology for selection among energy alternatives*, Expert Systems with Applications, Vol. 37, no. 9, pp. 6270–6281, 2014.

[9] LIU P., CHU Y., LI Y., et al. *Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making*, International Journal of Fuzzy Systems, Vol. 16, no. 2, pp. 242–255, 2014.

[10] LIU P., WANG Y., *Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean*, Neural Computing and Applications, Vol. 25, no. 7, pp. 2001–2010, 2014.

[11] MIAUMOTO M., TANAKA K., *Some properties of fuzzy sets of type 2*, Information and Control, Vol. 31, no. 4, pp. 312–340, 1976,

[12] NAYAGAM V., MURALIKRISHNAN S., SIVARAMAN G., *Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets*, Expert Systems with Applications, Vol. 38, no. 3, pp. 1464–1467, 2011.

[13] PEDRYCZ W., *Fuzzy sets in pattern recognition: methodology and methods*, Pattern Recognition, Vol. 23, no. 1, pp. 121–146, 1990.

[14] PENG J., WANG J., WANG J., et al. *An extension of ELECTRE to multi-criteria decision-making problems with multi-hesitant fuzzy sets*, Information Sciences, Vol. 307, no. 1, pp. 113–126, 2015.

[15] ROY B., *The outranking approach and the foundations of electre methods*, Theory and Decision, Vol. 31, no. 1, pp. 49–73, 1991.

[16] TORRA V., *Hesitant fuzzy sets*, International Journal of Intelligent Systems,Vol. 25, no. 6, pp. 529–539, 2010.

[17] VAHDANI B., HADIPOUR H., *Extension of the ELECTRE method based on interval-valued fuzzy sets*, Soft Computing, Vol. 15, no. 3, pp. 569–579, 2011.

[18] WANG J., NIE R., ZHANG H., et al. *New operators on triangular intuitionistic fuzzy numbers and their applications in system fault analysis*, Information Sciences, Vol. 251, no. 12, pp. 79–95, 2013.

[19] WANG J., WANG D., ZHANG H., et al. *Multi-criteria outranking approach with hesitant fuzzy sets*, OR Spectrum, Vol. 36, no. 4, pp. 1001–1019, 2014.

[20] WEI G., ZHAO X., WANG H., *An approach to multiple attribute group decision making with interval intuitionistic trapezoidal fuzzy information*, Technological and Economic Development of Economy, Vol. 18, no. 2, pp. 317–330, 2012.

[21] XIA M., XU Z., *Hesitant fuzzy information aggregation in decision making*, International Journal of Approximate Reasoning, Vol. 52, no. 3, pp. 395–407, 2011.

[22] YAGER R. R., *Multiple objective decision-making using fuzzy sets*, International Journal of Man-Machine Studies, Vol. 9, no. 4, pp. 375-C382, 1977.

[23] YAGER R. R., *ON THE THEORY OF BAGS*, International Journal of General Systems, Vol. 13, no. 1, pp. 23–37, 1986.

[24] ZADEH L A., *Fuzzy sets*, Information and Control, Vol. 8, no. 3, pp. 338–353, 1965.

[25] ZADEH L A., *Fuzzy logic and approximate reasoning*, Fuzzy sets, fuzzy logic, and fuzzy systems. World Scientific Publishing Co. Inc. pp. 238–259, 1996.

[26] ZADEH L A., *The concept of a linguistic variable and its application to approximate reasoning - II*, Learning Systems and Intelligent Robots. Springer US,pp. 199–249, 1974.

[27] ZHANG Z., *Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making*, Information Sciences, Vol. 234, no. 10, pp. 150–181, 2013.

[28] ZHANG N., WEI G., *Extension of VIKOR method for decision making problem based on hesitant fuzzy set*, Applied Mathematical Modelling, Vol. 37, no. 7, pp. 4938–4947, 2013.

[29] ZHU B., XU Z., *Extended hesitant fuzzy sets*, Technological and Economic Development of Economy, Vol. 22, no. 1, pp. 100–121, 2016.

[30] ZHU B., XU Z., XIA M., *Dual Hesitant Fuzzy Sets*, Journal of Applied Mathematics, Volume 2012, Article ID 879629, 13 pages, 2012.