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Invariant sets with arbitrary convex shapes in linear system dynamics

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Abstract A mathematical framework dedicated to flow-invariance study is proposed for continuous-time linear dynamics, with respect to general-shape contractive sets - defined by the class of proper C-sets (convex and compact sets, including the origin as an interior point), for which a constant-rate exponential decrease is considered. The main result provides a general algebraic characterization of flow-invariance, stated in terms of functions extending the concept of matrix measure (by using Minkowski functions, instead of vector norms, in the set description). A unifying point of view is created – able to connect several approaches to flow-invariance, separately reported in literature. This result is further exploited by set-embedding procedures that yield reformulations as optimization task – more reliable from the numerical perspective. Two examples are considered to illustrate how the flow-invariance criteria known for non-symmetrical polyhedrons, and symmetrical sets defined by weighted p-vector-norms, respectively, can be obtained as particular cases from our main result.

Key-words: continuous-time linear systems, invariant sets, Minkowski function, Lyapunov function.

1. Introduction

Consider a linear system with continuous-time dynamics described by the equation

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \qquad t \in \mathbb{R}_+,\ A \in \mathbb{R}^{n\times n}. \tag{1}$$

The paper focuses on the exploration of set invariance with respect to the trajectories of system (1), by constructing a framework that is able to accommodate invariant sets with arbitrary convex shapes. This construction relies on the properties of the Minkowski function associated with a compact and convex set, which allow the generalization of set-invariance criteria formulated in terms of matrix measures applied to operator $\boldsymbol{A}$. A basic, intuitive motivation is offered by the fact that the vector-norm-based description of a set is limited to symmetrical shapes, whereas the Minkowski-function-based description can accommodate non-symmetries.

The important steps in the mathematical development of flow-invariance theory for differential systems (with general form – *i.e.* nonlinear, non-autonomous) are summarized by the monograph [1]. In the control engineering literature, the first results were published during the eighties and early nineties; they are confined to linear dynamical systems, in continuous-time form (1) and discrete-time form

$$\begin{cases} x(t+1) = Ax(t) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \qquad t \in \mathbb{Z}_+,\ A \in \mathbb{R}^{n \times n}. \tag{2}$$

In 1984, paper [2] considers dynamics (1) and the invariant sets are symmetrical rectangles with arbitrary time-dependence; the constrained evolution of the trajectories is called "componentwise asymptotic stability" (the concept being later extended to dynamics (2), as well as to non-symmetrical rectangular invariant sets by [3], [4]). In 1988, paper [5] considers dynamics (2) and the invariant sets are time-constant, symmetrical polyhedrons. In 1990, paper [6] considers dynamics (1) and (2) and the invariant sets are exponentially-decreasing non-symmetrical polyhedrons; this concrete problem is placed within a general scenario that proposes the use of Minkowski functions for the description of invariant sets. In 1992, paper [7] considers dynamics (1) and (2), and points out the connection between general-form Lyapunov functions (the main goal of the work) and symmetrical invariant sets with arbitrary, symmetrical shapes described by weighted p-vector-norms.

All the aforementioned results provide concrete (numerically tractable) criteria for the characterization of invariant sets for systems (1) and (2), objective that was not targeted by the mathematical investigations, addressing the context of differential systems in general forms. Nevertheless, the statements and proofs of these results show that they were designed separately, and even in the nineties the subsequent developments of the respective directions did not take into account the identification of the connections (which would have been assumed to exist). A monograph on the role of set-invariance as analysis and design instrument for control engineering was elaborated by Blanchini and Miani in 2008 [8] and a revised edition in 2015 [9]. The approach does not offer a complete image of the connections between the research trends commented above, but it offers a comprehensive point of view for defining invariant sets, based on the Minkowski functions associated with *C-sets* (convex and compact sets, including the origin as an interior point) and contractive C-sets; on this background, the essential developments and discussions are oriented towards polyhedral sets. It is worth saying that the last decade brought noticeable contributions referring to the complexity of the studied dynamics (e.g. polytopic systems [9], [10], and switching systems [9], [11], [12]), as well as to the description of the invariant sets (relying on links to Laplace-Lyapunov equation [13], [14], [15]).

The framework proposed by this article intends to create a unifying presentation for the invariance of contractive C-sets with arbitrary shape, relative to the state trajectories of systems of form (1). Thus, our results can accommodate, as particular cases, most of the set-invariance criteria available in the literature for continuous-time systems (1). In addition, we must emphasize that our very recent work [16] is entirely dedicated to discrete-time dynamics of type (2) and ensures the study of invariant sets in similar terms, based on contractive C-sets.

The remainder of the current text is organized as follows. Section 2 provides a series of mathematical prerequisites needed by the rigorous presentation of the main result stated in section 3. Section 4 illustrates the possibility of particularizing the main result in order to obtain two set-invariance criteria already reported as individual methods, namely for the case of non-symmetrical polyhedrons in [6], and the case of symmetrical sets defined by weighted p-vector-norms in [7]; these particularizations are built by set-embedding procedures. Section 5 formulates some concluding remarks on the relevance of our research. An appendix includes the proofs of some technical results requested for the foundation of the key elements in our approach.

2. Notations and basic tools

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set that includes the origin as an interior point, $0 \in int(\mathcal{C})$. The Minkowski function (gauge) associated to $\mathcal{C}$ is given by

$$\nu_{\mathcal{C}} : \mathbb{R}^n \to \mathbb{R}_+, \;\; \nu_{\mathcal{C}}(x) = \inf \{\lambda > 0 | x \in \lambda\mathcal{C}\} \; , \tag{3}$$

The set $\mathcal{C}$ represents the unit ball corresponding to $\nu_{\mathcal{C}}$; the sublevel sets of $\nu_{\mathcal{C}}$ are obtained by linearly scaling the set $\mathcal{C}$. Some of the properties of $\nu_{\mathcal{C}}$ are presented next; see also, for instance, the properties presented by Proposition 3.12 in [9].

Proposition 1. ***(Properties of*** $\nu_{\mathcal{C}}$***)*** *Function* $\nu_{\mathcal{C}}$ *defined by* (3) *enjoys the following properties:*

a) It is positively homogeneous of order 1, i.e. $\nu_{\mathcal{C}}(\lambda x) = \lambda \nu_{\mathcal{C}}(x)$ *for all* $\lambda > 0$, $x \in \mathbb{R}^n$.

b) It is sub-additive, i.e. $\nu_{\mathcal{C}}(x_1 + x_2) \leq \nu_{\mathcal{C}}(x_1) + \nu_{\mathcal{C}}(x_2)$ *for all* $x_1, x_2 \in \mathbb{R}^n$.

c) It is convex, i.e. $\nu_{\mathcal{C}}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\nu_{\mathcal{C}}(x_1) + (1-\lambda)\nu_{\mathcal{C}}(x_2)$*, for all* $x_1,\ x_2 \in \mathbb{R}^n$ *and* $\lambda \in [0, 1]$.

d) It is Lipschitz continuous on $\mathbb{R}^n$.

e) $\nu_{\lambda\mathcal{C}}(x) = \nu_{\mathcal{C}}(\frac{1}{\lambda}x) = \frac{1}{\lambda}\nu_{\mathcal{C}}(x)$ *for all* $\lambda > 0$, $x \in \mathbb{R}^n$.

f) Let $\|\ \|$ *be a norm on* $\mathbb{R}^n$*. There exists* $\delta > 0$ *such that* $\nu_{\mathcal{C}}(x) \leq \delta \|x\|$ *for all* $x \in \mathbb{R}^n$.

g) If the set $\mathcal{C} \subset \mathbb{R}^n$ *is bounded, then there exists* $\gamma > 0$ *such that* $\nu_{\mathcal{C}}(x) \geq \gamma \|x\|$ *for all* $x \in \mathbb{R}^n$.

h) If $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subset \mathbb{R}^n$ *are convex sets so that* $0 \in int(\mathcal{C}_1)$*, then* $\nu_{\mathcal{C}_1}(x) \geq \nu_{\mathcal{C}_2}(x)$*, for all* $x \in \mathbb{R}^n$.

i) If the sets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n$ *are closed and convex sets so that* $0 \in int(\mathcal{C}_1 \cap \mathcal{C}_2)$ *and* $\nu_{\mathcal{C}_1}(x) \geq \nu_{\mathcal{C}_2}(x)$ *for all* $x \in \mathbb{R}^n$*, then* $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

j) If the set $\mathcal{C}$ *is symmetrical with respect to 0, i.e.* $x \in \mathcal{C} \Rightarrow -x \in \mathcal{C}$*, then* $\nu_{\mathcal{C}}(x) = \nu_{\mathcal{C}}(-x)$ *for all* $x \in \mathbb{R}^n$ *and, consequently,* $\nu_{\mathcal{C}}$ *represents a norm on* $\mathbb{R}^n$.

k) The set $\mathcal{C}$ *may be expressed as* $\mathcal{C} = \{x \in \mathbb{R}^n | \, \langle x^*, x\rangle \leq 1 \text{ for every } x^* \in \partial\nu_{\mathcal{C}}(0)\}$.

Based on the Minkowski function $\nu_{\mathcal{C}}$, we define two functions associated to a convex set $\mathcal{C}$ satisfying $0 \in int(\mathcal{C})$:

$$\varphi_{\mathcal{C}} : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}, \;\; \varphi_{\mathcal{C}}(Q) = \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Qx), \tag{4}$$

and

$$\eta_{\mathcal{C}} : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{+\infty\}, \;\; \eta_{\mathcal{C}}(Q) = \varphi'_{\mathcal{C}}(I_n, Q), \; = \lim_{\xi \searrow 0} \frac{1}{\xi} \left[\varphi_{\mathcal{C}}(I_n + \xi Q) - 1\right], \tag{5}$$

i.e. $\eta_{\mathcal{C}}(Q)$ represents the right directional derivative of $\varphi_{\mathcal{C}}$ at I_n in the direction of Q, where I_n denotes the identity matrix of order *n*.

These functions will be used for formulating our general results and their properties are detailed in the sequel.

Proposition 2. ***(Properties of** $\varphi_{\mathcal{C}}$**)*** *Function* $\varphi_{\mathcal{C}}$ *defined by* (4) *enjoys the following properties:*

a) $\varphi_{\mathcal{C}}(I_n) = 1$*,* $\varphi_{\mathcal{C}}(O_n) = 0$*,* $\varphi_{\mathcal{C}}(Q) \geq 0$ *for every* $Q \in \mathbb{R}^{n\times n}$*.*

b) $\varphi_{\mathcal{C}}$ *is sublinear on* $\mathbb{R}^{n\times n}$*, i.e. it is positively homogenous* $\varphi_{\mathcal{C}}(\lambda Q) = \lambda\varphi_{\mathcal{C}}(Q)$ *for any* $\lambda \geq 0$*, and subadditive:* $\varphi_{\mathcal{C}}(Q_1 + Q_2) \leq \varphi_{\mathcal{C}}(Q_1) + \varphi_{\mathcal{C}}(Q_2)$ *for any* $Q_1, Q_2 \in \mathbb{R}^{n\times n}$*.*

c) $\varphi_{\mathcal{C}}$ *is convex and continuous on* $\mathbb{R}^{n\times n}$*.*

d) If the set $\mathcal{C}$ *is compact (bounded and closed), then*

$$\varphi_{\mathcal{C}}(Q) = \sup_{x\neq 0} \frac{\nu_{\mathcal{C}}(Qx)}{\nu_{\mathcal{C}}(x)}\,. \tag{6}$$

e) If $F \in \mathbb{R}^{n\times n}$ *is such that* $\det(F) \neq 0$*, then*

$$\varphi_{F\mathcal{C}}(Q) = \varphi_{\mathcal{C}}(F^{-1}QF)\,. \tag{7}$$

f) For every $Q \in \mathbb{R}^{n\times n}$ *the following equality holds*

$$\varphi_{\mathcal{C}\cap(-\mathcal{C})}(-Q) = \varphi_{\mathcal{C}\cap(-\mathcal{C})}(Q)\,. \tag{8}$$

Proof. See the Appendix.

Proposition 3. ***(Properties of** $\eta_{\mathcal{C}}$**)*** *Function* $\eta_{\mathcal{C}}$ *defined by* (5) *enjoys the following properties:*

a) $\eta_{\mathcal{C}}(I_n) = 1$*,* $\eta_{\mathcal{C}}(O_n) = 0$*.*

b) $\eta_{\mathcal{C}}$ *is sublinear on* $\mathbb{R}^{n\times n}$*, i.e. it is positively homogenous* $\eta_{\mathcal{C}}(\lambda Q) = \lambda\eta_{\mathcal{C}}(Q)$ *for any* $\lambda \geq 0$*, and subadditive:* $\eta_{\mathcal{C}}(Q_1 + Q_2) \leq \eta_{\mathcal{C}}(Q_1) + \eta_{\mathcal{C}}(Q_2)$ *for any* $Q_1, Q_2 \in \mathbb{R}^{n\times n}$*.*

c) For every $Q \in \mathbb{R}^{n\times n}$ *the following statements hold:*

$$\eta_{\mathcal{C}}(Q + \alpha I_n) = \eta_{\mathcal{C}}(Q) + \alpha,\ for\ \ every\ \ \alpha \in \mathbb{R}; \tag{9}$$

$$\eta_{\mathcal{C}}(Q) \leq \varphi_{\mathcal{C}}(Q); \tag{10}$$

$$\eta_{\mathcal{C}}(Q) = \lim_{\theta\searrow 0}\frac{1}{\theta}\left[\varphi_{\mathcal{C}}(e^{\theta Q}) - 1\right]; \tag{11}$$

$$\eta_{\mathcal{C}}(Q) = \sup\left\{\langle x^*, Qx\rangle \,|\, x^* \in \partial\nu_{\mathcal{C}}(0), x \in \mathcal{C}, \langle x^*, x\rangle = 1\right\}, \tag{12}$$

where $\partial\nu_{\mathcal{C}}(0)$ *is the subdifferential of* $\nu_{\mathcal{C}}$ *at* $0 \in \mathbb{R}^n$ *and* $\langle a, b\rangle = a^T b$ *denotes the scalar product of the vectors* $a, b \in \mathbb{R}^n$*.*

d) If $F \in \mathbb{R}^{n\times n}$ *is such that* $\det(F) \neq 0$*, then*

$$\eta_{F\mathcal{C}}(Q) = \eta_{\mathcal{C}}(F^{-1}QF)\,. \tag{13}$$

Proof. See the Appendix.

Remark 1. Paper [17] introduces the matrix measures associated to a positive definite convex function $h : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mu_h : \mathbb{R}^{n\times n} \to \mathbb{R},\ \ \mu_h(Q) = \sup_{x\in\mathbb{R}^n, x\neq 0} \frac{h'(x, Qx)}{h(x)}\ , \tag{14}$$

and shows that if *h* is a vector norm on $\mathbb{R}^n$, *i.e.* $h(x) = \| x \|$, then $\mu_{\|\cdot\|}$ is the standard matrix measure induced by the norm $\|\cdot\|$ [18]. Following the procedure presented in [17], if the set $\mathcal{C}$ is compact and convex such that $0 \in int(\mathcal{C})$, then $\eta_{\mathcal{C}}$ defined by (5) is the matrix measure associated to the Minkowski function of $\mathcal{C}$, *i.e.* $\eta_{\mathcal{C}}(Q) = \mu_{\nu_{\mathcal{C}}}(Q)$ for every $Q \in \mathbb{R}^{n\times n}$.

Indeed, for every $Q \in \mathbb{R}^{n\times n}$ we get

$$\eta_{\mathcal{C}}(Q) = \varphi'_{\mathcal{C}}(I_n, Q) = \lim_{\xi\searrow 0} \frac{1}{\xi} \left[\sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(x + \xi Qx) - 1 \right] \tag{15}$$

Since $\nu_{\mathcal{C}}$ is positively homogeneous, then for every $y \in \mathbb{R}^n \backslash \{0\}$ we get $\nu_{\mathcal{C}}(\frac{1}{\nu_{\mathcal{C}}(y)} y) = 1$, therefore

$$\eta_{\mathcal{C}}(Q) \geq \lim_{\xi\searrow 0} \frac{1}{\xi} \left[\nu_{\mathcal{C}} \left(\frac{1}{\nu_{\mathcal{C}}(y)} y + \frac{1}{\nu_{\mathcal{C}}(y)} \xi Qy \right) - 1 \right]\ . \tag{16}$$

Consequently,

$$\begin{aligned} \eta_{\mathcal{C}}(Q) &\geq \sup_{y\in\mathbb{R}^n\backslash\{0\}} \lim_{\xi\searrow 0} \tfrac{1}{\xi} \left[\nu_{\mathcal{C}} \left(\tfrac{1}{\nu_{\mathcal{C}}(y)} y + \tfrac{1}{\nu_{\mathcal{C}}(y)} \xi Qy \right) - 1 \right] = \\ &= \sup_{y\in\mathbb{R}^n\backslash\{0\}} \tfrac{1}{\nu_{\mathcal{C}}(y)} \lim_{\lambda\searrow 0} \tfrac{1}{\lambda} \left[\nu_C(y + \lambda Qy) - \nu_C(y) \right] = \sup_{y\in\mathbb{R}^n\backslash\{0\}} \tfrac{\nu'_{\mathcal{C}}(y, Qy)}{\nu_{\mathcal{C}}(y)} = \mu_{\nu_{\mathcal{C}}}(Q). \end{aligned} \tag{17}$$

Conversely, since $\mathcal{C}$ is compact and $\varphi_{\mathcal{C}}$ is continuous on $\mathbb{R}^{n\times n}$ then for every sequence $\lambda_k \searrow 0$ there exists $\{x_k\}_{k\in\mathbb{N}} \subset bnd(\mathcal{C})$ such that $\varphi_{\mathcal{C}}(I + \lambda_k Q) = \nu_{\mathcal{C}}(x_k + \lambda_k Q x_k)$. Since the set $\{x_k\}_{k\in\mathbb{N}}$ is bounded then we find a subsequence $\{x_{k_p}\}_{p\in\mathbb{N}} \subset bnd(\mathcal{C})$ convergent to some $x_0 \in bnd(\mathcal{C})$.

By continuity we obtain

$$\begin{aligned} \eta_{\mathcal{C}}(Q) &= \varphi'_{\mathcal{C}}(I_n, Q) = \lim_{p\to\infty} \tfrac{1}{\lambda_{k_p}} \left[\varphi_{\mathcal{C}}(I_n + \lambda_{k_p} Q) - 1 \right] = \lim_{p\to\infty} \tfrac{1}{\lambda_{k_p}} \left[\nu_{\mathcal{C}}(x_{k_p} + \lambda_{k_p} Q x_{k_p}) - 1 \right] = \\ &= \lim_{p\to\infty} \tfrac{1}{\lambda_{k_p}} \left[\nu_{\mathcal{C}}(x_{k_p} + \lambda_{k_p} Q \lambda_{k_p}) - \nu_{\mathcal{C}}(x_{k_p}) \right] = \tfrac{\nu'_{\mathcal{C}}(x_0, Qx_0)}{\nu_{\mathcal{C}}(x_0)} \leq \sup_{x\in\mathbb{R}^n\backslash\{0\}} \tfrac{\nu'_{\mathcal{C}}(x, Qx)}{\nu_{\mathcal{C}}(x)} = \mu_{\nu_{\mathcal{C}}}(Q). \end{aligned} \tag{18}$$

3. Main results on invariance and stability supporting the general approach to flow-invariance

Let us consider the continuous-time linear system (1) defined with matrix $A \in \mathbb{R}^{n\times n}$. The main result of our paper, presented in the sequel, shows the role that functions (4) and (5) play in the characterization of convex sets that include the origin as an interior point, called C-sets, that are invariant with respect to the trajectories of system (1).

Theorem 1. *Let $\mathcal{C} \subset \mathbb{R}^n$ be a proper C-set and $\alpha \leq 0$. The following statements are equivalent:*

i) The matrix A of the continuous-time linear system (1) *satisfies the inequality*

$$\eta_{\mathcal{C}}(A) \le \alpha\,. \tag{19}$$

ii) The function

$$V : \mathbb{R}^n \to \mathbb{R}, \quad V(x) = \nu_{\mathcal{C}}(x), \tag{20}$$

is a Lyapunov function for system (1) *with the decreasing rate α along each trajectory, i.e.*

$$D^+V(x(t)) = \lim_{\theta\searrow 0} \tfrac{1}{\theta} \left[V(x(t+\theta)) - V(x(t))\right] \le \alpha V(x(t))\,, \forall t \in \mathbb{R}_+. \tag{21}$$

iii) The time-dependent set defined by

$$\mathcal{K} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n, \quad \mathcal{K}(t) = e^{\alpha t}\mathcal{C}\,, \tag{22}$$

is positively invariant with respect to the trajectories of system (1)*, i.e.*

$$\forall x_0 \in \mathcal{K}(0) \qquad \Rightarrow \qquad x(t) \in \mathcal{K}(t),\ \forall t \in \mathbb{R}_+, \tag{23}$$

where $x(t) = x(t; x_0)$ represents the trajectory of system (1) *initiated in $x(0) = x_0$.*

Proof. We first prove that for an arbitrary solution $x(t)$ to system (1), inequality (21) is equivalent to

$$V(x(t)) \le e^{\alpha(t-t^*)}V(x(t^*)), \ \ \forall t, t^* \in \mathbb{R}_+,\ t \ge t^*. \tag{24}$$

If (24) is satisfied, let $t^* \in \mathbb{R}_+$ be an arbitrary moment when we get

$$D^+V(x(t^*)) = \lim_{\theta\searrow 0} \frac{1}{\theta} \left[V(x(t^*+\theta)) - V(x(t^*))\right] \le V(x(t^*)) \lim_{\theta\searrow 0} \frac{e^{\alpha\theta}-1}{\theta} = \alpha V(x(t^*)), \tag{25}$$

therefore (21) is satisfied. Conversely, if (21) holds at $t^* \in \mathbb{R}_+$, then we consider the scalar differential equation $\dot{z}(t) = \alpha z(t)$ with the initial condition $z(t^*) = V(x(t^*))$. According to Theorem 4.2.11 in [19], for all $t \ge t^*$ we get $V(x(t)) \le z(t) = e^{\alpha(t-t^*)}z(t^*) = e^{\alpha(t-t^*)}V(x(t^*))$, therefore (24) holds.

(i) ⇒ (ii). If (19) is satisfied, using Proposition 3.(e), for an arbitrary solution $x(t)$ of system (1) we get

$$\begin{aligned} D^+V(x(t^*)) &= \lim_{\theta\searrow 0} \tfrac{1}{\theta} \left[\nu_{\mathcal{C}}(x(t^*+\theta)) - \nu_{\mathcal{C}}(x(t^*))\right] = \lim_{\theta\searrow 0} \tfrac{1}{\theta} \left[\nu_{\mathcal{C}}(e^{\theta A}x(t^*)) - \nu_{\mathcal{C}}(x(t^*))\right] \le \\ &\le \lim_{\theta\searrow 0} \tfrac{1}{\theta} \left[\varphi_{\mathcal{C}}(e^{\theta A})\nu_{\mathcal{C}}(x(t^*) - \nu_{\mathcal{C}}(x(t^*))\right] = \lim_{\theta\searrow 0} \tfrac{1}{\theta} \left[\varphi_{\mathcal{C}}(e^{\theta A}) - 1\right] \nu_{\mathcal{C}}(x(t^*) = \\ &= \eta_{\mathcal{C}}(x(t^*)\nu_{\mathcal{C}}(x(t^*) \le \alpha V(x(t^*)). \end{aligned} \tag{26}$$

(ii) ⇒ (iii). If (21) is satisfied along a trajectory $x(t)$ of system (1), then, using the equivalent form (24) we get that $\nu_{\mathcal{C}}(x(t)) \le e^{\alpha t}\nu_{\mathcal{C}}(x(0))$, $\forall t \in \mathbb{R}_+$. Taking an arbitrary initial condition $x(0) \in \mathcal{C}$, this leads to $\nu_{\mathcal{C}}(x(t)) \le e^{\alpha t}$ showing that $\nu_{\mathcal{K}(t)}(x(t)) = e^{-\alpha t}\nu_{\mathcal{C}}(x(t)) < 1$ therefore $x(t) \in \mathcal{K}(t)$. Thus $\mathcal{K}(t)$ is positively invariant with respect to system (1).

(iii) ⇒ (i). Consider an arbitrary initial condition on the boundary of $\mathcal{C}$, $x_0 \in bnd(\mathcal{C})$. Since $\mathcal{K}(t)$ is positively invariant with respect to system (1), the corresponding solution $x(t) = x(t; x_0)$ satisfies $\nu_{\mathcal{K}(t)}(x(t)) = e^{-\alpha t}\nu_{\mathcal{C}}(x(t)) = e^{-\alpha t}\nu_{\mathcal{C}}(e^{tA}x(0)) < 1 \Rightarrow \nu_{\mathcal{C}}(e^{tA}x(0)) < e^{\alpha t}$. Therefore

$$\varphi_{\mathcal{C}}(e^{tA}) = \sup_{x_0 \in bnd(\mathcal{C})} \nu_{\mathcal{C}}(e^{tA}x_0) \le e^{\alpha t} \tag{27}$$

and, consequently,

$$\eta_{\mathcal{C}}(A) = \lim_{\theta\searrow 0} \frac{1}{\theta} \left[\varphi_{\mathcal{C}}(e^{\theta A}) - 1\right] \le \lim_{\theta\searrow 0} \frac{1}{\theta} \left[e^{\alpha\theta} - 1\right] = \alpha\,. \tag{28}$$

Remark 2. In the particular case when the set $\mathcal{C} \subset \mathbb{R}^n$ is defined by $\nu_{\mathcal{C}}(x) = ||Px||_2$ with $P \in \mathbb{R}^{n\times n}$, $\det(P) \neq 0$, inequality (19) with $\alpha < 0$ in Theorem 1 leads to the well-known Lyapunov matrix inequality for continuous-time linear systems $A^TQ + QA \prec 0$, *i.e.* negative definite, where matrix $Q = P^TP \succ 0$ is positive definite. Indeed, once the set $\mathcal{C} \subset \mathbb{R}^n$ is defined by the 2-vector-norm, using the weighting square matrix $P \in \mathbb{R}^{n\times n}$, inequality (19) can be expressed in terms of the standard matrix measure relative to the induced matrix norm $|| \ ||_2$, namely $\mu_2(PAP^{-1}) = \lambda_{\max}\left(\frac{1}{2}(P^{-1})^TA^TP^T + \frac{1}{2}PAP^{-1}\right) \leq \alpha$, where $\lambda_{\max}(*)$ denotes the maximal eigenvalue of the symmetric matrix $*$. Consequently, the matrix $(P^{-1})^TA^TP^T + PAP^{-1} - 2\alpha I$ is negative semidefinite, meaning that $A^TP^TP + P^TPA - 2\alpha P^TP$ is also negative semidefinite. Hence, the matrix $A^T(P^TP) + (P^TP)A$ is negative definite.

4. Applications to particular cases of invariant sets

Let $m \in \mathbb{N}$, $m \geq n$, and consider a convex set $\Gamma \subset \mathbb{R}^m$ with $0 \in int(\Gamma)$. Let $G \in \mathbb{R}^{m\times n}$ be a matrix with $rank(G) = n$, so that the convex set

$$\mathcal{C} = \{x \in \mathbb{R}^n | Gx \in \Gamma\} \subset \mathbb{R}^n, \tag{29}$$

which is called *embedded* in set Γ, is also compact. Without loss of generality, we may assume that matrix *G* is partitioned as

$$G^T = \left[\begin{array}{cc} G_1^T & G_2^T \end{array}\right] \ with \ G_1 \in \mathbb{R}^{n\times n}, \det(G_1) \neq 0. \tag{30}$$

We intend to apply Theorem 1 in order to study the invariance of sets of form (29) with respect to system (1) for some particular cases which lead to results that were previously presented in literature.

To this end we consider the n dimensional subspace of $\mathbb{R}^m$

$$X = \{y = Gx \in \mathbb{R}^m | x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m. \tag{31}$$

Let

$$\mathcal{X} = \left\{\Phi \in \mathbb{R}^{m\times m} | Ker(\Phi) \supseteq X\right\} = \left\{\Phi \in \mathbb{R}^{m\times m} | \Phi G = O_{m\times n}\right\} \subseteq \mathbb{R}^{m\times m}, \tag{32}$$

where $Ker(\Phi) = \{y \in \mathbb{R}^m | \Phi y = 0\}$. Using matrix $A \in \mathbb{R}^{n\times n}$ of system (1) we build matrix

$$\widehat{A} = \left[\begin{array}{cc} GAG_1^{-1} & O_{m\times(m-n)} \end{array}\right] \in \mathbb{R}^{m\times m}. \tag{33}$$

The case when $m = n$ means $\det(G) \neq 0$ and $\widehat{A} = GAG^{-1}$.

The result to be presented below shows that upper bounds for $\eta_C(A)$ can be derived using the following dual optimization problems:

$$\psi = \inf_{\Phi\in\mathcal{X}} \sup_{\substack{y\in\Gamma \\ y^*\in\partial\nu_\Gamma(0) \\ \langle y^*,y\rangle=1}} \left\langle y^*, \left(\widehat{A} + \Phi\right) y\right\rangle \tag{34}$$

$$\overline{\psi} = \sup_{\substack{y\in\Gamma \\ y^*\in\partial\nu_\Gamma(0) \\ \langle y^*,y\rangle=1}} \inf_{\Phi\in\mathcal{X}} \left\langle y^*, \left(\widehat{A} + \Phi\right) y\right\rangle. \tag{35}$$

Theorem 2. *Let $\Gamma \subseteq \mathbb{R}^m$ be a convex set such that $0 \in int(\Gamma)$ and let matrix $G \in \mathbb{R}^{m\times n}$ satisfy condition* (30)*. For a C-set $\mathcal{C} \subset \mathbb{R}^n$ defined by* (29) *the following statements hold:*
a) $\eta_C(A) \le \psi$ and $\forall \Phi \in \mathcal{X}: \quad \eta_C(A) \le \eta_\Gamma(\widehat{A}+\Phi)$.
b) ($\eta_C(A) = \psi$ and $\exists \Phi \in \mathcal{X}: \quad \eta_C(A) = \eta_\Gamma(\widehat{A}+\Phi)$) if and only if $\overline{\psi} = \psi$.

Proof. See the Appendix.

4.1. Polyhedral sets

For $m \in \mathbb{N}^*$, $m \ge n$, let $\Gamma_m = \overline{1} - \mathbb{R}^m_+$ and consider $\Gamma = \Gamma_m \cap diag\{d_1, ..., d_m\}\Gamma_m$, where $\overline{1} = [1\ 1\ ...\ 1]^T \in \mathbb{R}^m$ and $d_i \in \{-1, 1\}$ for every $i \in \{1, ...m\}$. Let matrix $G \in \mathbb{R}^{m\times n}$ satisfy condition (30). We assume that the set $\mathcal{C} \subset \mathbb{R}^n$ defined by (29) is compact. In this case it has a polyhedral shape.

Corollary 1. *Let $\alpha \le 0$. Statements (i)–(iii) of Theorem 1 are true, if and only if there exists a matrix $H = [h_{ij}] \in \mathbb{R}^{m\times m}$ satisfying*

$$\begin{cases} HG = GA \\ \mu_{\|\|_\infty}(H) \le \alpha \\ h_{ij} \ge 0 \text{ for all } i \in I_+ = \{i \in \{1, ..., m\}\,|d_i = 1\} \text{ and all } \ j \ne i \end{cases} \tag{36}$$

Proof. For every $y = [y_1, ..., y_m]^T \in \mathbb{R}^m$ we get that

$$\nu_\Gamma(y) = \inf\{\lambda > 0|\ y \in \lambda\Gamma_m \text{ and } y \in \lambda D\Gamma_m\} = \max\{\nu_{\Gamma_m}(y), \nu_{D\Gamma_m}(y)\}. \tag{37}$$

Since $\nu_{\Gamma_m}(y) = \max\{0, y_1, ..., y_m\}$, then

$$\nu_\Gamma(y) = \max\{0, \max\{y_i|i \in I_+\}, \max\{|y_i|\ \ |i \in I_-\}\}, \tag{38}$$

where $I_- = \{i \in \{1, ..., m\}\,|d_i = -1\}$.

Now, for arbitrary $H \in \mathbb{R}^{m\times m}$, denote by $\theta_i = [h_{i1}...h_{im}]$ the i-th row of matrix H and compute

$$\begin{aligned} \varphi_\Gamma(H) &= \sup_{\nu_\Gamma(x)=1} \max\{0, \max\{|\theta_i x|\,|i \in I_-\}, \max\{\theta_i x|i \in I_+\}\} = \\ &= \begin{cases} +\infty, \ \exists i \in I_+ \text{ and } j \in \overline{1,m} \text{ such that } \ h_{ij} < 0, \\ \max\left\{\sum_{j=1}^m |h_{ij}|\,|\ \ i = \overline{1,m}\right\}, \text{ otherwise.} \end{cases} \end{aligned} \tag{39}$$

Taking the definition of η_{Γ_m} into account we obtain

$$\begin{aligned} \eta_{\Gamma_m}(H) &= \lim_{\lambda > 0} \tfrac{1}{\lambda}\left[\varphi_{\Gamma_m}(I_m + \lambda H) - 1\right] = \\ &= \begin{cases} +\infty, \text{ if } \exists i, j \in \overline{1,m}, \ i \ne j \text{ such that } \ h_{ij} < 0, \\ \max\left\{\lim_{\lambda > 0} \tfrac{1}{\lambda}\left[1 + \lambda\sum_{j=1}^n h_{ij} - 1\right] |i = \overline{1,m}\right\}, \text{otherwise.} \end{cases} \end{aligned} \tag{40}$$

After some calculations, the previous relation leads to

$$\eta_\Gamma(H) = \begin{cases} +\infty, \ \text{ if } \ \exists i \in I_+ \text{ and } j \in \overline{1,m}, j \ne i \text{ such that } h_{ij} < 0, \\ \max\left\{h_{ii} + \sum_{j=1, j\ne i}^m |h_{ij}|\,|i = \overline{1,m}\right\}, \text{ otherwise.} \end{cases} \tag{41}$$

From relation (38), if $\{e_1^*, ..., e_m^*\}$ is the canonical base in $\mathbb{R}^m$ we get

$$\partial\nu_\Gamma(0) = \text{conv}\left(\left\{e_i^*|i=\overline{1,m}\right\} \cup \{-e_i^*|i \in I_-\}\right), \tag{42}$$

where $conv(S)$ stands for the convex hull of a set $S \subset \mathbb{R}^n$, *i.e.* all the convex combinations of the elements of *S*. On the other hand, for every $i \in \{1, ...m\}$ we denote by $\gamma_i = [g_{i1}...g_{in}]$ the *i*-th row of matrix *G*, and compute

$$\overline{\psi}_i = \begin{cases} \sup\limits_{\substack{\gamma_i x=1 \\ \gamma_j x \le 1, j\in I_+\setminus\{i\} \\ |\gamma_k x|\le 1, k\in I_-}} \gamma_i Ax, & \text{if } i \in I_+ \\ \max\left\{ \sup\limits_{\substack{\gamma_i x=1 \\ \gamma_j x\le 1, j\in I_+ \\ |\gamma_k x|\le 1, k\in I_-\setminus\{i\}}} \gamma_i Ax, \sup\limits_{\substack{(-\gamma_i)x=-1 \\ (-\gamma_j)x\le 1, j\in I_+ \\ |(-\gamma_k)x|\le 1, k\in I_-\setminus\{i\}}} (-\gamma_i)Ax \right\}, & \text{if } i \in I_- \end{cases} \tag{43}$$

Using (35) leads to

$$\overline{\psi} = \sup_{\substack{y^*\in\partial\nu_\Gamma(0) \\ \langle y^*, Gx\rangle=1 \\ Gx\in\Gamma}} \langle y^*, GAx\rangle = \max_{i=\overline{1,m}} \overline{\psi}_i \tag{44}$$

For each of the linear programming problems appearing in (43) we consider the dual problem. Thus, for every $i \in \{1, ...m\}$ we get

$$\begin{aligned} &\sup_{\substack{\gamma_i x=1 \\ \gamma_j x\le 1, j\in I_+\setminus\{i\} \\ |\gamma_k x|\le 1, k\in I_-\setminus\{i\}}} \gamma_i Ax = \sup_{\substack{\gamma_i x=1 \\ \gamma_j x\le 1, j\ne i \\ -\gamma_k x\le 1, k\in I_-\setminus\{i\}}} \gamma_i Ax = \\ &= \sup_{\substack{[h_{i1}...h_{im}]G=\gamma_i A \\ h_{ij}\ge 0, j\ne i, j\in I_+ \\ h_{ik}=v_{ik}-w_{ik}, v_{ik}\ge 0, w_{ik}\ge 0, k\ne i, k\in I_-}} \left\{ h_{ii} + \sum_{j\in I_+, j\ne i} h_{ij} + \sum_{k\in I_-, k\ne i} (v_{ik}+w_{ik}) \right\} = \\ &= \sup_{\substack{[h_{i1}...h_{im}]G=\gamma_i A \\ h_{ij}\ge 0, j\ne i, j\in I_+}} \left\{ h_{ii} + \sum_{j\in I_+, j\ne i} h_{ij} + \sum_{k\in I_-, k\ne i} |h_{ik}| \right\} = \sup_{\substack{[h_{i1}...h_{im}]G=\gamma_i A \\ h_{ij}\ge 0, j\ne i, j\in I_+}} \left\{ h_{ii} + \sum_{j\ne i} |h_{ij}| \right\}. \end{aligned} \tag{45}$$

Thus we can build, line by line, a matrix $H \in \mathbb{R}^{m\times m}$ with $HG = GA$, with $h_{ij} \ge 0$ for all $i \in I_+$ and all $j \ne i$, satisfying $\mu_{\|\|_\infty}(H) = \eta_\Gamma(H) = \overline{\psi}$. Therefore $\overline{\psi} = \psi$ and the proof is completed.

Remark 3. (a) If in Corollary 1 we consider $d_i = 1$, $i \in \{1, ...m\}$, (*i.e.* $\Gamma = \Gamma_m$), then the set $\mathcal{C}$ defined by (29) can be equivalently described by

$$\mathcal{C} = \left\{x \in \mathbb{R}^n | Gx \le \overline{1}\right\} \subset \mathbb{R}^n, \tag{46}$$

where $\overline{1} = [1\ 1\ ...\ 1]^T \in \mathbb{R}^m$. Subsequently, statements (i) – (iii) of Theorem 1 are true if and only if there exists $H \in \mathbb{R}^{m\times m}$ with $HG = GA$, $h_{ij} \ge 0$ for $i \ne j$ and $H\overline{1} \le \alpha\overline{1}$. The proof is straightforward, relying on the expression of $\mu_{\|\|_\infty}(H)$.

This case means Theorem 4.33 in [9].

(b) If in Corollary 1 we consider $d_i = -1$, $i \in \{1, ...m\}$, (*i.e.* $\Gamma = \Gamma_m \cap \{-\Gamma_m\}$), then then the sets $\mathcal{C}$ defined by (29) are symmetrical and can be equivalently described by

$$C = \{x \in \mathbb{R}^n | \; \|Gx\|_\infty \leq 1\} \subset \mathbb{R}^n. \tag{47}$$

Subsequently, statements (i) – (iii) of Theorem 1 are true if and only if there exists $H \in \mathbb{R}^{m\times m}$ with $HG = GA$, and $\mu_{\|\|\|_\infty}(H) \leq \alpha$ (with no supplementary constraint on the nonnegativeness of h_{ij}).

This case is also discussed in [9]. There, for a matrix $H = [h_{ij}] \in \mathbb{R}^{m\times m}$ the following condition equivalent to $\mu_{\|\|\|_\infty}(H) \leq \alpha$ is considered

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix} \leq \alpha \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix}, \tag{48}$$

where matrices $H^+ = \left[h^+_{ij}\right]$, $H^- = \left[h^-_{ij}\right] \in \mathbb{R}^{m\times m}$ are defined as follows: $h^+_{ij} = \max\{h_{ij}, 0\}$ if $i \neq j$, $h^+_{ii} = h_{ii}$, and $h^-_{ij} = \max\{-h_{ij}, 0\}$ if $i \neq j$, $h^-_{ii} = 0$.

(c) If in Corollary 1 we consider only some $d_i = -1$ (not all of them), then the definition of the set $\mathcal{C}$ of form (29) includes some symmetrical conditions. Obviously, this case can be addressed relying on part (a) of the current Remark, if the set of inequalities in (46) includes two inequalities for each symmetrical condition. However this approach augments the number of rows in G, and, implicitly augments the size of unknown matrix H to be used in part (a) of Remark 3.

4.2. Symmetrical sets with arbitrary shapes

In $\mathbb{R}^m$ we consider the Hölder norm $\|\cdot\|_p$, $1 \leq p \leq \infty$. We apply Theorem 1 for the set $\Gamma = S_{\|\cdot\|_p}(0, 1) \subset \mathbb{R}^m$, *i.e.* the unit ball defined by $\|\cdot\|_p$, and a compact set $\mathcal{C} \subset \mathbb{R}^n$ defined by (29) with some matrix $G \in \mathbb{R}^{m\times n}$ satisfying (30). The set $\mathcal{C}$ is symmetrical, may have arbitrary shape, and can be equivalently described by

$$\mathcal{C} = \left\{x \in \mathbb{R}^n | \; \|Gx\|_p \leq 1\right\} \subset \mathbb{R}^n. \tag{49}$$

Corollary 2. *Let* $\alpha \leq 0$.

a) Statements (i) – (iii) of Theorem 1 are true if there exists a matrix $H \in \mathbb{R}^{m\times m}$ *satisfying*

$$\begin{cases} HG = GA \\ \mu_{\|\|\|_p}(H) \leq \alpha \end{cases} \tag{50}$$

b) Let $p \in \{1, \infty\}$. *Statements (i)–(iii) of Theorem 4.1 are true, if and only if there exists a matrix* $H \in \mathbb{R}^{m\times m}$ *satisfying* (50).

Proof. a) If there exists $H \in \mathbb{R}^{m\times m}$ with $HG = GA$and $\mu_{\|\cdot\|_p}(H) \leq \alpha$ then from Theorem 1 $\eta_{\mathcal{C}}(A) \leq \psi \leq \eta_\Gamma(H) = \mu_{\|\cdot\|_p}(H) \leq \alpha$ therefore statements (i) – (iii) of Theorem 1 are true.

The equivalence from b) follows immediately from Corollary 1. □

Remark 4. a) Corollary 2 includes, as particular cases, the results in [7], which do not take into consideration the concrete value α of the contraction rate of the invariant sets and the decreasing rate of the Lyapunov functions, respectively. Their results refer to a generic condition of form $\mu_{\|\cdot\|}(H) < 0$.

b) If the space $\mathbb{R}^m$, $m \geq n$, equipped with $\|\cdot\|$ has the "self-extension property" (see [20]), then part (a) of Corollary 2 becomes a necessary and sufficient condition. Part (b) of Corollary 5.3 represents a particular situation, since *inf*-norm and 1-norm ensure the "self-extension property". Generally speaking, the "self-extension property" is not necessary for the equality $\overline{\psi} = \psi$. We can prove that this equality is true if and only if for Γ and A there exists $\lambda_k \searrow 0$ and $H_k \in \mathbb{R}^{m\times m}$ with $H_k G = GA$ for every $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{1}{\lambda_k} \left[\varphi_\Gamma(I_m + \lambda_k H_k) - \varphi_C(I_n + \lambda_k A)\right] = 0. \tag{51}$$

It is obvious that (51) is satisfied if the space enjoys the "self-extension property".
c) If the considered norm is $\|\|_\infty$, then the sets defined by (47) and (49) are identical, and part b) of Corollary 2 is equivalent to Corollary 1 with $d_i = -1$, $i = \overline{1, m}$. □

5. Conclusions

The paper constructs a comprehensive scenario for the study of flow-invariance of continuous-time system (1), which considers general-shape contractive sets, described by Minkowski functions. The main result (Theorem 1) provides a general algebraic characterization of flow-invariance, expressed in terms of functions (x) that permit the extension of matrix measure properties. The key merit of this scenario consists in its unifying capabilities, which allow connections between several results previously reported as separate developments. Theorem 2 shows that, for large classes of sets, the embedding techniques allow reformulations as optimization problems. Corollaries 1 and 2 illustrate how the flow-invariance criteria known for non-symmetrical polyhedrons, and symmetrical sets defined by weighted *p*-vector-norms, respectively, can be obtained as particular cases from our main result.

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Appendix

Given a set $\mathcal{C} \subset \mathbb{R}^n$, its closure, interior and boundary of are denoted by $cl(\mathcal{C})$, $int(\mathcal{C})$ and $bnd(\mathcal{C}) = cl(\mathcal{C}) - int(\mathcal{C})$, respectively.

The *right directional derivative* of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ in the direction of $v \in \mathbb{R}^n$, $v \neq 0$, always exists and is given by

$$f'(x_0, v) = \lim_{\xi \searrow 0} \tfrac{1}{\xi} \left[f(x_0 + \xi v) - f(x_0) \right]. \tag{A-1}$$

The *subdifferential* of f at $x_0 \in \mathbb{R}^n$ is the set

$$\partial f(x_0) = \{x^* \in \mathbb{R}^n | \, \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0), \forall x \in \mathbb{R}^n\} \tag{A-2}$$

where $\langle \bullet, \bullet \rangle$ stands for the usual scalar product defined on $\mathbb{R}^n$ through $\langle v, x \rangle = v^T x$, for all $x, v \in \mathbb{R}^n$.

For arbitrary $\alpha \in \mathbb{R}$ we define the sets

$$[f = \alpha] = \{\mathrm{x} \in \mathbb{R}^n | f(\mathrm{x}) = \alpha\}, \quad [f \leq \alpha] = \{x \in \mathbb{R}^n | f(x) \leq \alpha\}. \tag{A-3}$$

The next result is used in the sequel (see [21]).

Lemma 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $x_0 \in \mathbb{R}^n$ be arbitrary.*
The following statements hold:
a) $\inf\limits_{x \in \mathbb{R}^n} f(x) < f(x_0)$ if and only if $\inf\limits_{v \in \mathbb{R}^n} f'(x_0, v) < 0$.
b) $f'(x_0, \cdot)$ is sublinear, i.e. positively homogenous and subadditive.
c) if $\alpha > \inf\limits_{x \in \mathbb{R}^n} f(x)$ then $[f \leq \alpha] = cl[f < \alpha]$.
d) for every $x, v \in \mathbb{R}^n$

$$\sup_{x^* \in \partial f(x)} \langle x^*, v \rangle = \lim_{\xi \searrow 0} \frac{1}{\xi} [f(x + \xi v) - f(x)] = f'(x, v). \tag{A-4}$$

e) if f is sublinear then for every $x_0 \in \mathbb{R}^n$

$$\partial f(x_0) = \{x^* \in \partial f(0) | \ \langle x^*, x_0 \rangle = f(x_0)\}. \tag{A-5}$$

Proof of Proposition 2. a) It is obvious from the definition of function $\varphi_{\mathcal{C}}$ (4) that $\varphi_{\mathcal{C}}(Q) \geq 0$ for every $Q \in \mathbb{R}^{n \times n}$; $\varphi_{\mathcal{C}}(I_n) = \sup\limits_{x \in bnd(\mathcal{C})} \nu_{\mathcal{C}}(x) = 1$, $\varphi_{\mathcal{C}}(O_n) = \nu_{\mathcal{C}}(0) = 0$.

b) For every $\alpha > 0$ and $Q \in \mathbb{R}^{n \times n}$ we have that

$$\varphi_{\mathcal{C}}(\alpha Q) = \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(\alpha Q x) = \alpha \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Qx) = \alpha \varphi_{\mathcal{C}}(Q). \tag{A-6}$$

For every $Q_1, Q_2 \in \mathbb{R}^{n \times n}$ we have that

$$\begin{aligned}\varphi_{\mathcal{C}}(Q_1 + Q_2) &= \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Q_1 x + Q_2 x) \leq \\ &\leq \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Q_1 x) + \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Q_2 x) = \varphi_{\mathcal{C}}(Q_1) + \varphi_{\mathcal{C}}(Q_2).\end{aligned} \tag{A-7}$$

c) For arbitrary $\lambda \in [0, 1]$ and Q_1, $Q_2 \in \mathbb{R}^{n \times n}$ we get

$$\begin{aligned}\varphi_{\mathcal{C}}(\lambda Q_1 + (1-\lambda) Q_2) &= \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}((\lambda Q_1 + (1-\lambda) Q_2) x) = \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(\lambda Q_1 x + (1-\lambda) Q_2 x) \leq \\ &\leq \sup_{\nu_{\mathcal{C}}(x)=1} [\nu_{\mathcal{C}}(\lambda Q_1 x) + \nu_{\mathcal{C}}((1-\lambda) Q_2 x)] \leq \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(\lambda Q_1 x) + \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}((1-\lambda) Q_2 x) = \\ &= \lambda \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Q_1 x) + (1-\lambda) \sup_{\nu_{\mathcal{C}}(x)=1} \nu_{\mathcal{C}}(Q_2 x) = \lambda \varphi_{\mathcal{C}}(Q_1) + (1-\lambda) \eta_{\mathcal{C}}(Q_2),\end{aligned} \tag{A-8}$$

showing that $\varphi_{\mathcal{C}}$ is a convex function on $\mathbb{R}^{n \times n}$.

For every $Q \in \mathbb{R}^{n \times n}$, if $\{Q_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^{n \times n}$, $Q_k \to Q$ it follows

$$\begin{aligned}|\varphi_{\mathcal{C}}(Q_k) - \varphi_{\mathcal{C}}(Q)| &\leq \sup_{\nu_{\mathcal{C}}(x)=1} \max\{\nu_{\mathcal{C}}(Q_k x - Qx), \nu_{\mathcal{C}}(Qx - Q_k x)\} \leq \\ &\leq \delta \sup_{\nu_{\mathcal{C}}(x)=1} \|Qx - Q_k x\| \leq \delta \|Q - Q_k\| \quad \text{for all } k \in \mathbb{N},\end{aligned} \tag{A-9}$$

henceφ_C is continuous on $\mathbb{R}^{n\times n}$.

d) Let the set $\mathcal{C}\subset\mathbb{R}^n$ compact. Then, properties f) and g) from Proposition 1 lead to the existence of $\delta,\gamma>0$ so that $\gamma\|x\|\le\nu_{\mathcal{C}}(x)\le\delta\|x\|$ for all $x\in\mathbb{R}^n$. If $x\neq 0$, then $||x||>0$ implying $\nu_{\mathcal{C}}(x)>0$ for all $x\in\mathbb{R}^n\backslash\{0\}$, that allows the following evaluation

$$\varphi_{\mathcal{C}}(Q)=\sup_{\nu_{\mathcal{C}}(x)=1}\nu_{\mathcal{C}}(Qx)=\sup_{\nu_{\mathcal{C}}(x)\neq 0}\nu_{\mathcal{C}}\left(Q\frac{x}{\nu_{\mathcal{C}}(x)}\right)=\sup_{x\neq 0}\frac{\nu_{\mathcal{C}}(Qx)}{\nu_{\mathcal{C}}(x)}. \tag{A-10}$$

e) If $F\in\mathbb{R}^{n\times n}$ is such that $\det(F)\neq 0$, then for every $Q\in\mathbb{R}^{n\times n}$,

$$\begin{aligned}\varphi_{F\mathcal{C}}(Q)&=\sup_{\nu_{F\mathcal{C}}(x)=1}\nu_{F\mathcal{C}}(Qx)=\sup_{\nu_{\mathcal{C}}(F^{-1}x)=1}\nu_{\mathcal{C}}(F^{-1}Qx)=\sup_{\nu_{\mathcal{C}}(y)=1}\nu_{\mathcal{C}}(F^{-1}QFy)=\\&=\varphi_{\mathcal{C}}(F^{-1}MF).\end{aligned} \tag{A-11}$$

f) For every $Q\in\mathbb{R}^{n\times n}$

$$\begin{aligned}\varphi_{\mathcal{C}\cap(-\mathcal{C})}(-Q)&=\sup_{\nu_{\mathcal{C}\cap(-\mathcal{C})}(x)=1}\nu_{\mathcal{C}\cap(-\mathcal{C})}(-Qx)=\sup_{\nu_{\mathcal{C}\cap(-\mathcal{C})}(x)=1}\nu_{\mathcal{C}\cap(-\mathcal{C})}(Qx)=\\&=\varphi_{\mathcal{C}\cap(-\mathcal{C})}(Q),\end{aligned} \tag{A-12}$$

since the set $\mathcal{C}$ is symmetrical with respect to 0 and $0\in\mathcal{C}\cap(-\mathcal{C})$.

Proof of Proposition 3. a) The following evaluations are immediate:

$$\eta_{\mathcal{C}}(I_n)=\lim_{\xi\searrow 0}\tfrac{1}{\xi}[\varphi_{\mathcal{C}}(I_n+\xi I_n)-1]=\lim_{\xi\searrow 0}\tfrac{1}{\xi}[1+\xi-1]=1\,, \tag{A-13}$$

$$\eta_{\mathcal{C}}(O_n)=\lim_{\xi\searrow 0}\tfrac{1}{\xi}[\varphi_{\mathcal{C}}(I_n+\xi O_n)-1]=\lim_{\xi\searrow 0}\tfrac{1}{\xi}[1-1]=0\,. \tag{A-14}$$

b) It follows immediately from the definition of the function $\eta_{\mathcal{C}}$ and from Lemma 1 (b).

c) For every $\alpha\in\mathbb{R}$ and for every $Q\in\mathbb{R}^{n\times n}$ we get

$$\begin{aligned}&\eta_{\mathcal{C}}(Q+\alpha I_n)=\lim_{\xi\searrow 0}\tfrac{1}{\xi}\left[\varphi_{\mathcal{C}}(I_n+\xi(Q+\alpha I_n))-1\right]=\lim_{\xi\searrow 0}\tfrac{1}{\xi}\left[\varphi_{l\mathcal{C}}((1+\xi\alpha)I_n+\xi Q)-1\right]\\&=\lim_{\xi\searrow 0}\tfrac{1+\xi\alpha}{\xi}\left[\varphi_{\mathcal{C}}(I_n+\tfrac{\xi}{1+\xi\alpha}Q)-\tfrac{1}{1+\xi\alpha}\right]\overset{\zeta=\frac{\xi}{1+\xi\alpha}}{=}\lim_{\zeta\searrow 0}\tfrac{1}{\zeta}\left[\varphi_{\mathcal{C}}(I_n+\zeta Q)-1+\zeta\alpha\right]=\eta_{\mathcal{C}}(Q)+\alpha.\end{aligned} \tag{A-15}$$

d) Since $\varphi_{\mathcal{C}}$ is sublinear, then using the biconjugate theorem we obtain

$$\begin{aligned}\eta_{\mathcal{C}}(Q)&=\sup_{\substack{Q^*\in\ \partial\varphi_{\mathcal{C}}(O_n)\\ \langle Q^*,I_n\rangle=1}}\langle Q^*,Q\rangle=\sup_{\substack{Q^*\in\mathbb{R}^{n\times n}\\ \langle Q^*,I_n\rangle=1}}\langle Q^*,Q\rangle-I_{\partial\varphi_{\mathcal{C}}(O_n)}=\\&=\sup_{\substack{Q^*\in\mathbb{R}^{n\times n}\\ \langle Q^*,I_n\rangle=1}}\langle Q^*,Q\rangle-\varphi_C^*(Q^*)\le\sup_{Q^*\in\mathbb{R}^{n\times n}}\langle Q^*,Q\rangle-\varphi_C^*(Q^*)=\varphi_C^{**}(Q)=\varphi_C(Q),\end{aligned} \tag{A-16}$$

where $\varphi_C^*(Q^*)=\sup_{Q\in\mathbb{R}^{n\times n}}[\langle Q^*,Q\rangle-\varphi_C(Q)]$ is the conjugate function of $\varphi_{\mathcal{C}}$.

e) Let $\theta > 0$ and write $e^{\theta Q} = I_n + \theta Q + \theta O(\theta)$, where $O(\theta) \in \mathbb{R}^{n\times n}$ satisfies $\lim_{\theta\searrow 0} O(\theta) = 0$. The sublinearity of $\varphi_{\mathcal{C}}$ leads to

$$\varphi_{\mathcal{C}}(I_n + \theta Q) - \theta\varphi_{\mathcal{C}}(O(\theta)) \le \varphi_{\mathcal{C}}(I_n + \theta Q + \theta O(\theta)) \le \varphi_{\mathcal{C}}(I_n + \theta Q) + \theta\varphi_{\mathcal{C}}(O(\theta))\,, \quad \text{(A-17)}$$

equivalent to

$$\tfrac{1}{\theta}\varphi_{\mathcal{C}}(I_n + \theta Q) - \varphi_{\mathcal{C}}(O(\theta)) \le \tfrac{1}{\theta}\varphi_{\mathcal{C}}(e^{\theta Q}) \le \tfrac{1}{\theta}\varphi_{\mathcal{C}}(I_n + \theta Q) + \varphi_{\mathcal{C}}(O(\theta))\,, \quad \text{(A-18)}$$

which further implies

$$\lim_{\theta\searrow 0}\tfrac{1}{\theta}\left[\varphi_{\mathcal{C}}(e^{\theta Q}) - 1\right] = \lim_{\theta\searrow 0}\tfrac{1}{\theta}\left[\varphi_{\mathcal{C}}(I_n + \theta Q) - 1\right] = \eta_{\mathcal{C}}(Q)\,. \quad \text{(A-19)}$$

f) For the beginning we point out that for $x \in \mathcal{C}$ the condition $\langle x^*, x\rangle = 1$ with $x^* \in \partial\nu_{\mathcal{C}}(0)$ is equivalent to $x \in bnd(\mathcal{C})$ since $1 \ge \nu_{\mathcal{C}}(x) \ge \langle x^*, x\rangle = 1$.

First, for every $Q \in \mathbb{R}^{n\times n}$ and for every $x \in \mathcal{C}$, $x^* \in \partial\nu_{\mathcal{C}}(0)$ with $\langle x^*, x\rangle = 1$ we have

$$\begin{array}{rcl}\eta_{\mathcal{C}}(M) & = & \lim_{\xi\searrow 0}\tfrac{1}{\xi}\left[\varphi_{\mathcal{C}}(I_n + \xi Q) - 1\right] \ge \lim_{\xi\searrow 0}\tfrac{1}{\xi}\left[\nu_{\mathcal{C}}(x + \xi Q x) - 1\right] \\ & \ge & \lim_{\xi\searrow 0}\tfrac{1}{\xi}\left[\langle x^*, x + \xi Q x\rangle - 1\right] = \langle x^*, Qx\rangle \end{array} . \quad \text{(A-20)}$$

therefore

$$\sup\{\,\langle x^*, Qx\rangle \,|\, x^* \in \partial\nu_{\mathcal{C}}(0),\ x \in \mathcal{C},\ \langle x^*, x\rangle = 1\} \le \eta_{\mathcal{C}}(Q)\,. \quad \text{(A-21)}$$

Next we prove the converse statement. Let us denote

$$\gamma = \sup\{\langle x^*, Qx\rangle\,|x^* \in \partial\nu_{\mathcal{C}}(0), x \in \mathcal{C}, \langle x^*, x\rangle = 1\}\ . \quad \text{(A-22)}$$

If $\gamma = \infty$ then it is obvious that $\eta_{\mathcal{C}}(Q) \le \gamma$.

For $\gamma \in \mathbb{R}$ we prove that $\eta_{\mathcal{C}}(Q) \le \gamma$ by contradiction. Assuming that $\eta_{\mathcal{C}}(Q) > \gamma$, there exists $\gamma' \in \mathbb{R}$ so that $\eta_{\mathcal{C}}(Q) > \gamma' > \gamma$. Based on the continuity of $\varphi_{\mathcal{C}}$ on $\mathbb{R}^{n\times n}$, there exists $\lambda_0 > 0$ such that $\frac{1}{\lambda}\left[\varphi_{\mathcal{C}}(I_n + \lambda M) - 1\right] > \gamma'$ for every $\lambda \in (0, \lambda_0]$. Therefore we find $x_\lambda \in \mathcal{C}$ with $\nu_{\mathcal{C}}(x_\lambda) = 1$ such that $\frac{1}{\lambda}\left[\nu_{\mathcal{C}}(x_\lambda + \lambda Q x_\lambda) - 1\right] > \gamma'$. Taking into account that $\nu_{\mathcal{C}}$ is convex and continuous, from the mean theorem we find $\bar{x}_\lambda \in [x_\lambda, x_\lambda + \lambda Q x_\lambda]$ and $\bar{x}^*_\lambda \in \partial\nu_{\mathcal{C}}(\bar{x}_\lambda)$ such that $\langle \bar{x}^*_\lambda, Q\bar{x}_\lambda\rangle = \frac{1}{\lambda}\left[\nu_{\mathcal{C}}(x_\lambda + \lambda Q x_\lambda) - \nu_{\mathcal{C}}(x_\lambda)\right] > \gamma'$ for every $\lambda \in (0, \lambda_0]$. Since $\partial\nu_{\mathcal{C}}(0)$ is compact and $\bar{x}^*_\lambda \in \partial\nu_C(\bar{x}_\lambda)$ if and only if $\bar{x}^*_\lambda \in \partial\nu_{\mathcal{C}}(0)$ and $\langle \bar{x}^*_\lambda, \bar{x}_\lambda\rangle = \nu_{\mathcal{C}}(x_\lambda) = 1$ then $\gamma = \sup\{\,\langle x^*, Qx\rangle \,|\, x^* \in \partial\nu_{\mathcal{C}}(0), x \in \mathcal{C}, \langle x^*, x\rangle = 1\} \ge \gamma' > \gamma$ which cannot be true.

Therefore $\eta_{\mathcal{C}}(Q) \le \gamma$ and the proof is completed.

g) If $\det(F) \neq 0$ then there exists F^{-1} and taking Proposition 2 (e) into account we obtain

$$\eta_{F\mathcal{C}}(Q) = \lim_{\xi\searrow 0}\frac{1}{\xi}\left[\varphi_{F\mathcal{C}}(I_n + \xi Q) - 1\right] = \lim_{\xi\searrow 0}\frac{1}{\xi}\left[\varphi_{\mathcal{C}}(I_n + \xi F^{-1} Q F) - 1\right] = \eta_{\mathcal{C}}(F^{-1} Q F). \quad \text{(A-23)}$$

Proof of Theorem 2. a) Let $\Phi \in \mathcal{X}$. From Proposition 3(f) we get that

$$\eta_{\mathcal{C}}(A) = \sup\{\,\langle x^*, Ax\rangle \,|\ x^* \in \partial\nu_{\mathcal{C}}(0), x \in \mathcal{C}, \langle x^*, x\rangle = 1\}\,. \quad \text{(A-24)}$$

First we prove that

$$\partial \nu_{\mathcal{C}}(0) = \left\{ x^* = G^T y^* \middle| \;\; y^* \in \partial \nu_\Gamma(0) \right\}. \tag{A-25}$$

Indeed, if $x^* \in \partial \nu_{\mathcal{C}}(0)$ then

$$\nu_{\mathcal{C}}(x) \geq \langle x^*, x \rangle \;\; \text{for every} \;\; x \in X. \tag{A-26}$$

Since $\nu_{\mathcal{C}}(x) = \nu_\Gamma(Gx)$ for every $x \in \mathbb{R}^n$, from Hahn-Banach Theorem we find that there exists $y^* \in \mathbb{R}^m$ such that

$$\nu_\Gamma(y) \geq \langle y^*, y \rangle \;\; \text{for every} \;\; y \in \mathbb{R}^m, \;\; \text{and} \;\; \langle x^*, x \rangle = \langle y^*, Gx \rangle \;\; \text{for every} \; x \in \mathbb{R}^n \tag{A-27}$$

therefore $y^* \in \partial \nu_\Gamma(0)$ and $x^* = G^T y^*$. Conversely, if $y^* \in \partial \nu_\Gamma(0)$ then

$$\left\langle G^T y^*, x \right\rangle = \langle y^*, Gx \rangle \leq \nu_\Gamma(Gx) = \nu_C(x), \tag{A-28}$$

thus $G^T y^* \in \partial \nu_C(0)$ and (21) is proved.

Next, let $\Phi \in \mathcal{X}$ be arbitrary. Taking (A-25) into account we get

$$\begin{aligned}
\eta_{\mathcal{C}}(A) &= \sup \left\{ \left\langle G^T y^*, Ax \right\rangle \middle| \;\; y^* \in \partial \nu_\Gamma(0), Gx \in \Gamma, \langle y^*, Gx \rangle = 1 \right\} = \\
&= \sup \left\{ \left\langle y^*, \widehat{A} y + \Phi y \right\rangle \middle| \;\; y^* \in \partial \nu_\Gamma(0), y \in \Gamma \cap X, \langle y^*, y \rangle = 1 \right\} \leq \\
&\leq \sup \left\{ \left\langle y^*, \left(\widehat{A} + \Phi\right) y \right\rangle \middle| \;\; y^* \in \partial \nu_\Gamma(0), y \in \Gamma, \langle y^*, y \rangle = 1 \right\} = \eta_\Gamma(\widehat{A} + \Phi).
\end{aligned} \tag{A-29}$$

Consequently,

$$\eta_{\mathcal{C}}(A) \leq \inf_{\Phi \in \mathcal{X}} \eta_\Gamma\left(\widehat{A} + \Phi\right) = \inf_{\Phi \in \mathcal{X}} \sup_{\substack{y \in \Gamma \\ y^* \in \partial \nu_\Gamma(0) \\ \langle y^*, y \rangle = 1}} \left\langle y^*, \left(\widehat{A} + \Phi\right) y \right\rangle = \psi. \tag{A-30}$$

b) To prove part b) it suffices to show that

$$\overline{\psi} = \eta_{\mathcal{C}}(A). \tag{A-31}$$

Taking into account that $\mathcal{X}$ is a linear subspace of $\mathbb{R}^{m \times m}$, if $y \notin X$ then $\inf_{\Phi \in \mathcal{X}} \left\langle y^*, \left(\widehat{A} + \Phi\right) y \right\rangle = -\infty$. Therefore, by simple calculations we obtain

$$\begin{aligned}
\overline{\psi} &= \sup \left\{ \left\langle y^*, \widehat{A} y \right\rangle \middle| \;\; y^* \in \partial \nu_\Gamma(0), y \in \Gamma \cap X, \langle y^*, y \rangle = 1 \right\} = \\
&= \sup \left\{ \langle y^*, GAx \rangle \middle| \;\; y^* \in \partial \nu_\Gamma(0), Gx \in \Gamma, \langle y^*, Gx \rangle = 1 \right\}.
\end{aligned} \tag{A-32}$$

From (A-25), for every $x \in \mathcal{C}$ we have that $\partial \nu_{\mathcal{C}}(x) = G^T(\partial \nu_\Gamma(Gx))$ therefore

$$\begin{aligned}
\overline{\psi} &= \sup_{\substack{y^* \in \partial \nu_\Gamma(0) \\ \langle y^*, Gx \rangle = 1 \\ Gx \in \Gamma}} \langle y^*, GAx \rangle = \sup_{\substack{y^* \in \partial \nu_\Gamma(Gx) \\ x \in bnd(\mathcal{C})}} \left\langle G^T y^*, Ax \right\rangle = \\
&= \sup_{\substack{x^* \in \partial \nu_{\mathcal{C}}(x) \\ x \in bnd(\mathcal{C})}} \langle x^*, Ax \rangle = \sup_{\substack{x^* \in \partial \nu_{\mathcal{C}}(0) \\ \langle x^*, x \rangle = 1 \\ x \in \mathcal{C}}} \langle x^*, Ax \rangle = \eta_{\mathcal{C}}(A).
\end{aligned} \tag{A-33}$$

This completes the proof.