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State Estimation of Permanent Magnet Synchronous Motor Dynamics Using Higher-order Continuous-Discrete Filtering Equations

Ravish H. HIRPARA

S.N. Patel Institute of Technology and Research Centre, Umrakh, Bardoli, Gujarat, India
E-mails: ravishhirpara@gmail.com, ravish.hirpara@snpitrc.ac.in

Abstract. In electrical machine theory the *permanent magnet synchronous motor* (PMSM) dynamics has been studied extensively. In this paper we have derived filtering equations of PMSM dynamics using higher-order continuous-discrete filter and simulated these equations with two sets of initial values. In this paper process noise is involved with continuous state dynamics of the PMSM and that is coupled with discrete measurement equation, which is involved with discrete measurement noise. Notably, the filtering equations of this paper explicit contribution of the predicted estimate and the observation noise to filtered estimate. The filtering theory presented in this paper will be useful to applied mathematicians, control engineers for better understanding of the continuous-discrete filtering equation of the non-linear dynamics, especially for stochastic problems arising from intelligent industrial systems.

Key-words: Continuous-discrete filtering equations; non-linear filtering; PMSM dynamics; stochastic differential equations; stochastic systems.

1. Introduction

In the classification of the dynamical systems, the deterministic systems are noise free dynamics, these systems can be represent in *Ordinary Differential Equations* (ODEs) and stochastic systems are noisy dynamics, these systems can be represent in *Stochastic Differential Equations* (SDEs). The stochastic dynamics in combination with the observation equation is becomes filtering model. Whenever observations are not easily available from the dynamics then for the state estimation the *Fokker-Planck equation* (FPE) is useful tool, as shown in [1]–[3], and for Differential flatness theory of the FPE, for instance, in [4]. Whenever noisy observation available, stochastic dynamics coupled with observation equation becomes stochastic filtering problem and to estimate their states stochastic filtering theory is useful tool. The permanent magnet synchronous motor has striking application in industrial systems. In such application it is important to estimates state of dynamical systems for better operation. Estimation of unmeasurable state variables is commonly called observation. A device (or a computer program) that estimates or observes the states is called a state-estimator [5]. State estimation is a procedure to estimate the state variables of a system by eliminating inaccuracies and errors from measurement data. For the deterministic systems Luenberger state observer is useful tool. The PMSM dynamics is highly non-linear system and when process noise involved in the system that becomes stochastic system. On other hand when ever noisy observations available that becomes non-linear filtering problem. For linear stochastic systems, linear filtering is required for the state estimation of the system by using *Kalman Filtering* (KF) and for non-linear stochastic systems non-linear filtering is required for the state estimation of the system by using *Extended Kalman Filtering* (EKF). For higher-order continuous-continuous non-linear filtering [6], [7]. In this paper we have considered the PMSM continuous dynamics has non-linear state equations are coupled with discrete linear observation equations. The papers [8] to [13] treat the continuous-discrete EKF and for accurate continuous-discrete EKF in [14], for higher-order accurate continuous-discrete EKF [15]. The nonlinear prediction by separation approach is discussed in [16] and for differential flatness theory and derivation-free nonlinear Kalman filtering the [17] is useful. Various other relevant approaches for various applications are iterative feedback tuning in fuzzy control systems [18], hybrid particle filter-particle swarm optimization algorithm and application to fuzzy controlled servo systems [19], analysis and design of a high efficiency current mode buck converter with I2C controlled output voltage [20], aspects concerning the observation process modelling in the framework of cognition processes [21], system identification using long short term memory recurrent neural networks for real time conical tank system [22], and tensor product-based model transformation approach to tower crane systems modeling [23]. For more non-linearity involved in the system the author has developed higher-order continuous-discrete filtering. Higher-order approaches may give better results whenever system nonlinearities are especially severe. These higher-order approaches include second-order filtering, iterated Kalman filtering, sum based Kalman filtering, and grid-based Kalman filtering. These filters provide ways to reduce the linearization errors that are part of the extended Kalman Filtering. They usually give estimation performance that is better than the extended Kalman filtering, but they are, a lot more complicated, and they required a lot more computer time.

With these motivations, this paper develops and analyses continuous-discrete non-linear filtering equations for the two-phase PMSM system, which leads to It SDEs. The term continuous-discrete filtering of this paper accounts for the process noise in the dynamical equation and the observation noise in the discrete measurement equation. The filtering theory of this paper adopts the discrete measurement system. The discrete measurement system gives rise to the notion of a two-stage estimation procedure. The two-stage estimation procedure accounts prediction equations between the observations and filtering equations at the observation instants. As this world move towards the digital world this tool will be useful to understand intelligent industrial systems with example of the state estimation of the two-phase PMSM system.

This paper is organized as follows: Section 2 describes two-phase PMSM dynamics higher-order continuous-discrete filtering. Section 3 offers numerical results. Concluding remarks are given in Section 4.

2. Two-phase PMSM Dynamics and Higher-order Continuous-discrete Filtering

Let us consider the continuous state dynamics and discrete observation filtering model:

$$dx_t = a(x_t, t)dt + b(x_t, t)dB_t, \tag{1}$$

$$y_{t_k} = h(x_{t_k}, t_k) + v_{t_k}, \tag{2}$$

where x_t and $a(x_t, t)$ denote the state vector and system non-linearity respectively. The term $b(x_t, t)$ is the process noise coefficient and the term B_t is Brownian motion process with $var(dB_t) = Idt$ Furthermore, the terms $y_{t_k}, h(x_{t_k}, t_k)$ indicate the observation vector and the measurement non-linearity respectively. The observation noise is normally distributed v_{t_k} is $N(0, \eta_k)$, where the term η_k is a real-valued observation noise variance matrix. The proposed higher-order filtering, *e.g.* second-order continuous-discrete filtering equations, has two parts: first "between the observations" and secondly, "at the observation". For a greater detail about the proposed higher-order filtering equations available in [7] and [24]. The design of discrete second-order filters for continuous-discrete models is treated in [25] as well.

In between the observations, the mean and variance evolutions are

$$\langle x_i(\tau)\rangle = (a_i(\langle x_\tau\rangle, \tau) + \frac{1}{2}\sum_{p,q} P_{pq}\partial_{\langle x_p\rangle\langle x_q\rangle} a_i(\langle x_\tau\rangle, \tau))d\tau, \tag{3}$$

$$\begin{aligned} dP_{ij}(\tau) = \Big(&\sum_p P_{ip}\partial_{\langle x_p\rangle} a_j(\langle x_\tau\rangle, \tau) + \sum_p P_{jp}\partial_{\langle x_p\rangle} a_i(\langle x_\tau\rangle, \tau\Big) \\ &+ (bb^T)_{ij}(\langle x_\tau\rangle, \tau) + \frac{1}{2}\sum_{p,q} P_{pq}\partial_{\langle x_p\rangle\langle x_q\rangle}(bb^T)_{ij}(\langle x_\tau\rangle, \tau))d\tau \end{aligned} \tag{4}$$

where
$\langle x_i(\tau)\rangle = E(x_i(\tau)\,|Y_{t_{k-1}}),$
$P_{ij}(\tau) = E((x_i(\tau) - E(x_i(\tau)\,|Y_{t_{k-1}}))(x_j(\tau) - E(x_j(\tau)\,|Y_{t_{k-1}}))\,|Y_{t_{k-1}}),$
$\partial_{\langle x_p\rangle\langle x_q\rangle} = \frac{\partial^2}{\partial\langle x_p\rangle\partial\langle x_q\rangle}, \partial_{\langle x_p\rangle} = \frac{\partial}{\partial\langle x_p\rangle}.$

The operator E denotes the conditional expectation operator given the accumulated observation $Y_{t_{k-1}} = \{y_{t_n}, 0 \le n \le k-1\}, t_{k-1} \le \tau \le t_k$ and t_{k-1}, t_k are observation instants.

At the observation,

$$\begin{aligned} \langle x_i^{t_k}\rangle = \left\langle x_i^{t_{k-1}}\right\rangle + (&\sum_p P_{ip}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k))(\sum_{p,q} P_{pq}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) \\ &\partial_{\left\langle x_q^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) + \frac{1}{2}\sum_{p,q,r,s} P_{pr}P_{qs}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle\left\langle x_q^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) \\ &\times\partial_{\left\langle x_r^{t_{k-1}}\right\rangle\left\langle x_s^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) + \eta)^{-1} \times (y_{t_k} - h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) - \\ &\frac{1}{2}\sum_{p,q} P_{pq}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle\left\langle x_q^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)), \end{aligned} \tag{5}$$

$$P_{ij}^{t_k} = P_{ij}^{t_{k-1}} - ((\sum_p P_{ip}\partial_{\langle x_p^{t_{k-1}}\rangle} h^T(\langle x_{t_k}^{t_{k-1}}\rangle, t_k)) \times (\sum_{p,q} P_{pq}\partial_{\langle x_p^{t_{k-1}}\rangle} h(\langle x_{t_k}^{t_{k-1}}\rangle, t_k)$$
$$\partial_{\langle x_q^{t_{k-1}}\rangle} h^T(\langle x_{t_k}^{t_{k-1}}\rangle, t_k) + \frac{1}{2}\sum_{p,q,r,s} P_{pr}P_{qs}\partial_{\langle x_p^{t_{k-1}}\rangle\langle x_q^{t_{k-1}}\rangle} h(\langle x_{t_k}^{t_{k-1}}\rangle, t_k)$$
$$\times\partial_{\langle x_r^{t_{k-1}}\rangle\langle x_s^{t_{k-1}}\rangle} h^T(\langle x_{t_k}^{t_{k-1}}\rangle, t_k) + \eta)^{-1} \times (\sum_p P_{jp}\partial_{\langle x_p^{t_{k-1}}\rangle} h(\langle x_{t_k}^{t_{k-1}}\rangle, t_k))), \quad (6)$$

where
$\langle x_i^{t_k}\rangle = E(x_i(t_k)\,|Y_{t_k}), \langle x_i^{t_{k-1}}\rangle = E(x_i(t_k)\,|Y_{t_{k-1}}),$
$P_{ij}^{t_k} = E((x_i(t_k) - E(x_i(t_k)\,|Y_t))(x_j(t_k) - E(x_j(t_k)\,|Y_{t_k}))\,|Y_{t_k},$
$P_{ij}^{t_{k-1}} = E((x_i(t_k) - E(x_i(t_k)\,|Y_{t_{k-1}}))(x_j(t_k) - E(x_j(t_k)\,|Y_{t_{k-1}}))\,|Y_{t_{k-1}},$
$\partial_{\langle x_p^{t_{k-1}}\rangle} = \frac{\partial}{\partial\langle x_p^{t_{k-1}}\rangle}, \partial_{\langle x_q^{t_{k-1}}\rangle} = \frac{\partial}{\partial\langle x_q^{t_{k-1}}\rangle}, \partial_{\langle x_p^{t_{k-1}}\rangle\langle x_q^{t_{k-1}}\rangle} = \frac{\partial^2}{\partial\langle x_p^{t_{k-1}}\rangle\partial\langle x_q^{t_{k-1}}\rangle},$
$\partial_{\langle x_r^{t_{k-1}}\rangle\langle x_s^{t_{k-1}}\rangle} = \frac{\partial^2}{\partial\langle x_r^{t_{k-1}}\rangle\partial\langle x_s^{t_{k-1}}\rangle}.$

Let us consider the following model for a two-phase permanent magnet synchronous motor, where i_α and i_β are the currents in the two windings, θ and ω are the angular position and velocity of the rotor, R and L are the winding resistance and inductance, λ is the flux constant, and F is the coefficient of viscous friction. The control inputs u_α and u_β consist of the applied voltages across the two windings, and J is the moment of inertia of the motor shaft and load [9], [10]. The state is defined as

$$di_\alpha = (-\frac{R}{L}i_\alpha + \frac{\omega\lambda}{L}sin\theta + \frac{u_\alpha}{L})dt + \frac{\sigma_1}{L}dB_1, \quad (7)$$

$$di_\beta = (-\frac{R}{L}i_\beta - \frac{\omega\lambda}{L}cos\theta + \frac{u_\beta}{L})dt + \frac{\sigma_2}{L}dB_2, \quad (8)$$

$$d\omega = (-\frac{3\lambda}{2J}i_\alpha sin\theta + \frac{3\lambda}{2J}i_\beta cos\theta - \frac{F}{J}\omega)dt + \sigma_3 dB_3, \quad (9)$$

$$d\theta = \omega dt, \quad (10)$$

The vector terms B_1, B_2, B_3 are Brownian process noise due to uncertainty in the control inputs and the load torque and terms $\sigma_1, \sigma_2, \sigma_3$ are state independent process noise coefficient. After consider $x_t = (x_1, x_2, x_3, x_4)^T = (i_\alpha, i_\beta, \omega, \theta)^T$ and $u_t = (u_1, u_2)^T = (u_\alpha, u_\beta)^T = (sin2\pi ft, cos2\pi ft)^T$ and combining with (7) and comparing with (1), the result is

$$dx_1 = (-\frac{R}{L}x_1 + \frac{\lambda x_3}{L}sin\,x_4 + \frac{u_\alpha}{L})dt + \frac{\sigma_1}{L}dB_1, \quad (11)$$

$$dx_2 = (-\frac{R}{L}x_2 - \frac{\lambda x_3}{L}cos\,x_4 + \frac{u_\beta}{L})dt + \frac{\sigma_2}{L}dB_2, \quad (12)$$

$$dx_3 = (-\frac{3\lambda}{2J}x_1\,sin\,x_4 + \frac{3\lambda}{2J}x_2\,cos\,x_4 - \frac{F}{J}x_3)dt + \sigma_3 dB_3, \quad (13)$$

$$dx_4 = x_3 dt, \tag{14}$$

The above system can be re-stated as

$$a(x_t,t) = (a_i(x_t,t))_{1\le i\le 4} = \begin{pmatrix} -\frac{R}{L}x_1 + \frac{\lambda x_3}{L} sin\, x_4 + \frac{u_\alpha}{L} \\ -\frac{R}{L}x_2 - \frac{\lambda x_3}{L} cos\, x_4 + \frac{u_\beta}{L} \\ -\frac{3\lambda}{2J}x_1 sin\, x_4 + \frac{3\lambda}{2J}x_2 cos\, x_4 - \frac{F}{J}x_3 \\ x_3 \end{pmatrix},$$

$$b(x_t,t) = (b_i(t))_{1\le i\le 4} = \begin{pmatrix} \frac{\sigma_1}{L} \\ \frac{\sigma_2}{L} \\ \sigma_3 \\ 0 \end{pmatrix}, dB_t = \begin{pmatrix} dB_1 \\ dB_2 \\ dB_3 \\ 0 \end{pmatrix}.$$

Supposing that the winding currents can be measured with sense resistors, the measurement equations are

$$y_{1t_k} = x_{1t_k} + v_{1t_k}, \tag{15}$$

$$y_{2t_k} = x_{2t_k} + v_{2t_k}, \tag{16}$$

where $y_{t_k} = (y_{1t_k}, y_{2t_k})^T$ is the observation vector, $v_{t_k} = (v_{1t_k}, v_{2t_k})^T$ is the noise vector, the measurement nonlinearity is $h(x_{t_k}, t_k) = (h_1(x_{t_k}, t_k),\ h_2(x_{t_k}, t_k))^T$ is $(x_{1t_k}, x_{2t_k})^T$, and observation noise variance is $\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$.

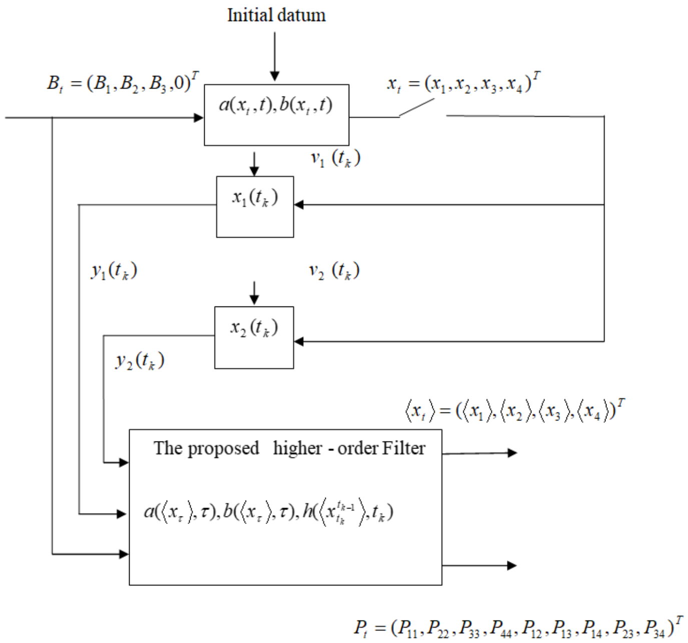

Fig. 1. Block diagram of the proposed higher-order filter.

Figure 1 shows the block diagram representation of the proposed higher-order filter. The proposed higher-order filtering algorithm for continuous dynamics coupled with the discrete observations is shown in Figure 2. Prediction step means “between the observations” and update step means “at the observations”.

Algorithm **Continuous-discrete proposed higher-order filter**

Input: $m_0, P_0, \{y_k\}_{k-1}^{T}$

for $k = 1.........T$ **do**

Prediction step (Between the observations)

Solve for $m(t_k)$ and $P(t_k)$

$$\frac{dm}{dt} = a(m,t) + \frac{1}{2} PA_{xx}(m,t)$$

$$m(t_{k-1}) = m_{k-1}$$

$$\frac{dP}{dt} = A_x(m,t)P + PA_x(m,t)^T + b(m,t)b(m,t)^T$$

$$+ \frac{1}{2} PB_{xx}(m,t)B_{xx}(m,t)^T$$

$$P(t_{k-1}) = P_{k-1}$$

$$m_k^- = m(t_k)$$

$$P_k^- = P(t_k)$$

Update Step (At the observations)

$$W_k = y_k - h_k(m_k^-) - \frac{1}{2} P_k^- [H_k]_{xx}(m_k^-, P_k^-)$$

$$S_k = [H_k]_x(m_k^-, P_k^-)P_k^- [[H_k]_x(m_k^-, P_k^-)]^T$$

$$+ [H_k]_{xx}(m_k^-, P_k^-)P_k^{-2} [[H_k]_{xx}(m_k^-, P_k^-)]^T + \eta_k$$

$$K_k = P_k^- [[H_k]_x(m_k^-, P_k^-)]^T S_k^{-1}$$

$$m_k = m_k^- + K_k W_k$$

$$P_k = P_k^- - K_k S_k K_k^T$$

end for

Fig. 2. Algorithm for the proposed higher-order filtering.

In Figure 2, A_x, H_x are first order partials and A_{xx}, H_{xx}, B_{xx} are second-order partials. Making use of (1), (3) and (4), the mean and variance evolutions for the two-phase PMSM dynamics are obtained:

$$d\langle x_1\rangle = (-\frac{R}{L}\langle x_1\rangle + \frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle + \frac{u_\alpha}{L} - P_{44}\frac{\langle x_3\rangle\lambda}{2L}sin\langle x_4\rangle + P_{34}\frac{\lambda}{L}cos\langle x_4\rangle)d\tau, \quad (17)$$

$$d\langle x_2\rangle = (-\frac{R}{L}\langle x_2\rangle - \frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle + \frac{u_\beta}{L} + P_{44}\frac{\langle x_3\rangle\lambda}{2L}cos\langle x_4\rangle + P_{34}\frac{\lambda}{L}sin\langle x_4\rangle)d\tau, \quad (18)$$

$$\begin{aligned}\langle x_3\rangle = (-\frac{3\lambda}{2J}\langle x_1\rangle sin\langle x_4\rangle + \frac{3\lambda}{2J}\langle x_2\rangle cos\langle x_4\rangle - \frac{F}{J}\langle x_3\rangle \\ +\frac{P_{44}}{2}(\frac{3\lambda}{2J}\langle x_1\rangle sin\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle cos\langle x_4\rangle) - P_{14}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{24}\frac{3\lambda}{2J}sin\langle x_4\rangle)d\tau, \quad (19)\end{aligned}$$

$$d\langle x_4\rangle = \langle x_3\rangle d\tau, \quad (20)$$

$$dP_{11} = (2(-P_{11}\frac{R}{L} + P_{13}\frac{\lambda}{L}sin\langle x_4\rangle + P_{14}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle) + \frac{\sigma_1^2}{L^2})d\tau, \quad (21)$$

$$dP_{22} = (2(-P_{22}\frac{R}{L} - P_{23}\frac{\lambda}{L}cos\langle x_4\rangle + P_{24}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle) + \frac{\sigma_2^2}{L^2})d\tau, \quad (22)$$

$$\begin{aligned}dP_{33} = (2(-P_{13}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{23}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{33}\frac{F}{J} \\ +P_{34}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle)) + \sigma_3^2)d\tau, \quad (23)\end{aligned}$$

$$dP_{44} = 2P_{34}d\tau, \quad (24)$$

$$\begin{aligned}dP_{12} = dP_{21} = (-P_{12}\frac{R}{L} - P_{13}\frac{\lambda}{L}cos\langle x_4\rangle + P_{14}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle - P_{12}\frac{R}{L} \\ +P_{23}\frac{\lambda}{L}sin\langle x_4\rangle + P_{24}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle + \frac{\sigma_1\sigma_2}{L^2})d\tau, \quad (25)\end{aligned}$$

$$\begin{aligned}dP_{13} = dP_{31} = (-P_{11}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{12}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{13}\frac{F}{J} \\ +P_{14}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle) \\ -P_{13}\frac{R}{L} + P_{23}\frac{\lambda}{L}sin\langle x_4\rangle + P_{34}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle + \frac{\sigma_1\sigma_3}{L})d\tau, \quad (26)\end{aligned}$$

$$dP_{14} = dP_{41} = (P_{13} - P_{14}\frac{R}{L} + P_{34}\frac{\lambda}{L}sin\langle x_4\rangle + P_{44}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle)d\tau, \quad (27)$$

$$dP_{23} = dP_{32} = (-P_{12}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{22}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{23}\frac{F}{J}$$
$$+P_{24}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle)$$
$$-P_{23}\frac{R}{L} - P_{33}\frac{\lambda}{L}cos\langle x_4\rangle + P_{34}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle + \frac{\sigma_2\sigma_3}{L})d\tau, \quad (28)$$

$$dP_{24} = dP_{42} = (P_{23} - P_{24}\frac{R}{L} - P_{34}\frac{\lambda}{L}cos\langle x_4\rangle + P_{44}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle)d\tau, \quad (29)$$

$$dP_{34} = dP_{43} = (P_{33} - P_{14}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{24}\frac{3\lambda}{2J}cos\langle x_4\rangle$$
$$-P_{34}\frac{F}{J} + P_{44}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle))d\tau, \quad (30)$$

Equations (17)-(30) are system of equations, *i.e.* (17)-(20) and (21)-(30). The right-hand term $\langle x_i(\tau)\rangle$ of equations (17)-(20) denotes $\langle x_i(\tau)\rangle = E(x_i(\tau)\left|Y_{t_{k-1}}\right.)$.

The right-hand term $P_{ij}(\tau)$ of equations (21)-(30) denotes $P_{ij}(\tau) = E((x_i(\tau)-E(x_i(\tau)\left|Y_{t_{k-1}}\right.))(x_j(\tau)-E(x_j(\tau)\left|Y_{t_{k-1}}\right.))\left|Y_{t_{k-1}}\right.)$, where $1 \leq i \leq 4, 1 \leq j \leq 4$. After combining (2) with (5) and (6), this leads to

$$\left\langle x_i^{t_k}\right\rangle = \left\langle x_i^{t_{k-1}}\right\rangle + ((P_{i1}^{t_{k-1}}(P_{22}^{t_{k-1}} + \eta) - P_{i2}^{t_{k-1}}P_{12}^{t_{k-1}})(y_{t_k}^1 - \left\langle x_{1t_k}^{t_{k-1}}\right\rangle)$$
$$+(P_{i2}^{t_{k-1}}(P_{11}^{t_{k-1}} + \eta)$$
$$-P_{i1}^{t_{k-1}}P_{12}^{t_{k-1}})(y_{t_k}^2 - \left\langle x_{2t_k}^{t_{k-1}}\right\rangle))/D, \quad (31)$$

$$P_{ij}^{t_k} = P_{ij}^{t_{k-1}} - ((P_{i1}^{t_{k-1}}(P_{22}^{t_{k-1}} + \eta) - P_{i2}^{t_{k-1}}P_{12}^{t_{k-1}})(P_{j1}^{t_{k-1}}) + (P_{i2}^{t_{k-1}}(P_{11}^{t_{k-1}} + \eta)$$
$$-P_{i1}^{t_{k-1}}P_{12}^{t_{k-1}})(P_{j2}^{t_{k-1}}))/D, \quad (32)$$

where $D = (P_{11}^{t_{k-1}} + \eta)(P_{22}^{t_{k-1}} + \eta) - {P_{12}^{t_{k-1}}}^2$.

The above system describes the conditional mean and variance evolutions at the observation. The right-hand side term $\langle x_i\rangle$ of equation (31) denotes $\left\langle x_i^{t_k}\right\rangle = E(x_i(t_k)\left|Y_{t_{k-1}}\right.)$. The right-hand side term P_{ij} of equation (32) denotes $P_{ij}^{t_k} = E((x_i(t_k) - E(x_i(t_k)\left|Y_{t_{k-1}}\right.))(x_j(t_k) - E(x_j(t_k)\left|Y_{t_{k-1}}\right.))\left|Y_{t_{k-1}}\right.)$.

Continuous-discrete *Extended Kalman Filtering* (EKF) for the two-phase PMSM dynamics. The *extended Kalman filter* (EKF) is a non-linear filter that accounts for the first-order partials of system non-linearity and measurement non-linearity. The practical applications are autonomous systems, non-linear dynamic circuits and satellite orbit determination problems. The whole analysis can be divided in two parts: first "between the observation" and secondly, "at the observation".

In between the observation, the conditional mean and variance evolutions are

$$d\langle x_i(\tau)\rangle = a_i(\langle x_\tau\rangle, \tau)d\tau, \tag{33}$$

$$dP_{ij}(\tau) = (\sum_p P_{ip}\partial_{\langle x_p\rangle} a_j(\langle x_\tau\rangle, \tau) + \sum_p P_{jp}\partial_{\langle x_p\rangle} a_i(\langle x_\tau\rangle, \tau) + (bb^T)_{ij}(\tau))d\tau \tag{34}$$

where

$\langle x_i(\tau)\rangle = E(x_i(\tau)|Y_{t_{k-1}})P_{ij}(\tau) = E((x_i(\tau) - E(x_i(\tau)\left|Y_{t_{k-1}}\right.))(x_j(\tau) - E(x_j(\tau)\left|Y_{t_{k-1}}\right.))\left|Y_{t_{k-1}}\right.)$.

The operator E denotes the conditional expectation operator given the accumulated observation $Y_{t_{k-1}} t_{k-1} \le \tau \le t_k$ and t_{k-1}, t_k are observation instants and where $1 \le i \le 4, 1 \le j \le 4$.

At the observation,

$$\begin{aligned}\langle x_i^{t_k}\rangle = \left\langle x_i^{t_{k-1}}\right\rangle + (\sum_p P_{ip}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)\\ \times(\sum_{p,q} P_{pq}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)\partial_{\left\langle x_q^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) + \eta)^{-1}\\ \times(y_{t_k} - h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)),\end{aligned} \tag{35}$$

$$\begin{aligned}P_{ij}^{t_k} = P_{ij}^{t_{k-1}} - ((\sum_p P_{ip}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k))\\ \times(\sum_{p,q} P_{pq}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)\partial_{\left\langle x_q^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) + \eta)^{-1}\\ \times(\sum_p P_{jp}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)).\end{aligned} \tag{36}$$

After combining (1) with (33) and (34), the result is

$$d\langle x_1\rangle = (-\frac{R}{L}\langle x_1\rangle + \frac{\langle x_3\rangle\lambda}{L} sin\langle x_4\rangle + \frac{u_\alpha}{L})d\tau, \tag{37}$$

$$d\langle x_2\rangle = (-\frac{R}{L}\langle x_2\rangle - \frac{\langle x_3\rangle\lambda}{L} cos\langle x_4\rangle + \frac{u_\beta}{L})d\tau, \tag{38}$$

$$d\langle x_3\rangle = (-\frac{3\lambda}{2J}\langle x_1\rangle sin\langle x_4\rangle + \frac{3\lambda}{2J}\langle x_2\rangle cos\langle x_4\rangle - \frac{F}{J}\langle x_3\rangle)d\tau, \tag{39}$$

$$d\langle x_4\rangle = \langle x_3\rangle d\tau, \tag{40}$$

$$dP_{11} = (2(-P_{11}\frac{R}{L} + P_{13}\frac{\lambda}{L} sin\langle x_4\rangle + P_{14}\frac{\langle x_3\rangle\lambda}{L} cos\langle x_4\rangle) + \frac{\sigma_1^2}{L^2})d\tau, \tag{41}$$

$$dP_{22} = (2(-P_{22}\frac{R}{L} - P_{23}\frac{\lambda}{L} cos\langle x_4\rangle + P_{24}\frac{\langle x_3\rangle\lambda}{L} sin\langle x_4\rangle) + \frac{\sigma_2^2}{L^2})d\tau, \tag{42}$$

$$dP_{33} = (2(-P_{13}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{23}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{33}\frac{F}{J} + P_{34}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle)) + \sigma_3^2)d\tau, \quad (43)$$

$$dP_{44} = 2P_{34}d\tau, \quad (44)$$

$$dP_{12} = dP_{21} = (-P_{12}\frac{R}{L} - P_{13}\frac{\lambda}{L}cos\langle x_4\rangle + P_{14}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle - P_{12}\frac{R}{L} + P_{23}\frac{\lambda}{L}sin\langle x_4\rangle + P_{24}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle + \frac{\sigma_1\sigma_2}{L^2})d\tau, \quad (45)$$

$$dP_{13} = dP_{31} = (-P_{11}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{12}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{13}\frac{F}{J} + P_{14}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle) - P_{13}\frac{R}{L} + P_{23}\frac{\lambda}{L}sin\langle x_4\rangle + P_{34}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle + \frac{\sigma_1\sigma_3}{L})d\tau, \quad (46)$$

$$dP_{14} = dP_{41} = (P_{13} - P_{14}\frac{R}{L} + P_{34}\frac{\lambda}{L}sin\langle x_4\rangle + P_{44}\frac{\langle x_3\rangle\lambda}{L}cos\langle x_4\rangle)d\tau, \quad (47)$$

$$dP_{23} = dP_{32} = (-P_{12}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{22}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{23}\frac{F}{J} + P_{24}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle) - P_{23}\frac{R}{L} - P_{33}\frac{\lambda}{L}cos\langle x_4\rangle + P_{34}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle + \frac{\sigma_2\sigma_3}{L})d\tau, \quad (48)$$

$$dP_{24} = dP_{42} = (P_{23} - P_{24}\frac{R}{L} - P_{34}\frac{\lambda}{L}cos\langle x_4\rangle + P_{44}\frac{\langle x_3\rangle\lambda}{L}sin\langle x_4\rangle)d\tau, \quad (49)$$

$$dP_{34} = dP_{43} = (P_{33} - P_{14}\frac{3\lambda}{2J}sin\langle x_4\rangle + P_{24}\frac{3\lambda}{2J}cos\langle x_4\rangle - P_{34}\frac{F}{J} + P_{44}(-\frac{3\lambda}{2J}\langle x_1\rangle cos\langle x_4\rangle - \frac{3\lambda}{2J}\langle x_2\rangle sin\langle x_4\rangle))d\tau. \quad (50)$$

After combining (2) with (35) and (36), the result is

$$\left\langle x_i^{t_k}\right\rangle = \left\langle x_i^{t_{k-1}}\right\rangle + ((P_{i1}^{t_{k-1}}(P_{22}^{t_{k-1}} + \eta) - P_{i2}^{t_{k-1}}P_{12}^{t_{k-1}})(y_{t_k}^1 - \left\langle x_{1t_k}^{t_{k-1}}\right\rangle) + (P_{i2}^{t_{k-1}}(P_{11}^{t_{k-1}} + \eta) - P_{i1}^{t_{k-1}}P_{12}^{t_{k-1}})(y_{t_k}^2 - \left\langle x_{2t_k}^{t_{k-1}}\right\rangle))/D, \quad (51)$$

$$P_{ij}^{t_k} = P_{ij}^{t_{k-1}} - ((P_{i1}^{t_{k-1}}(P_{22}^{t_{k-1}} + \eta) - P_{i2}^{t_{k-1}}P_{12}^{t_{k-1}})(P_{j1}^{t_{k-1}}) + (P_{i2}^{t_{k-1}}(P_{11}^{t_{k-1}} + \eta) \\ P_{i1}^{t_{k-1}}P_{12}^{t_{k-1}})(P_{j2}^{t_{k-1}}))/D, \quad (52)$$

where $D = (P_{11}^{t_{k-1}} + \eta)(P_{22}^{t_{k-1}} + \eta) - {P_{12}^{t_{k-1}}}^2$.

Equations (17)(30) and (31)-(32) describe the continuous-discrete filtering equations for the two-phase PMSM dynamics that are derived the first time.

3. Numerical Simulation Results

This section examines the efficacy of the estimation algorithm of the paper by accomplishing the digital experimentation that involves two different sets of initial data. Here, the following initial conditions, system parameters as well as the initial variances are chosen [5]:

$$R = 1.5\Omega, L = 3mH, \lambda = 0.1Vs/rad, J = 0.002\ Kg/m^2,$$

$$F = 0.001, \langle x_1(0)\rangle = 0.5\ Amp, \langle x_2(0)\rangle = 0.5\ Amp,$$

$$\langle x_3(0)\rangle = 0.1\ rad/sec, \langle x_4(0)\rangle = 0.2\ rad,$$

$$\sigma_1 = 0.001, \sigma_2 = 0.001, \sigma_3 = 0.05, \eta = 0.5,$$

$$P_{11}(0) = 1\ Amp^2, P_{22}(0) = 1\ Amp^2, P_{33}(0) = 1\ rad^2/sec^2,$$

$$P_{44}(0) = 1\ rad^2, P_{ij}(0), i \neq j = 0.$$

Figures 3 to 6 give numerical simulations results of the true and filtered trajectories of the states of the two-phase PMSM dynamics. The dotted line denotes the true trajectories of all states. The solid line denotes higher-order 'filtered' state trajectories. The extended Kalman filtering trajectories are displayed via the solid line and higher-order filtering equations are demonstrated in Figures 7 to 10 via the dotted line. In this paper, the filter efficacy is adjudged on the basis of a relatively less intensity of the observation noise as well as the larger intensity.

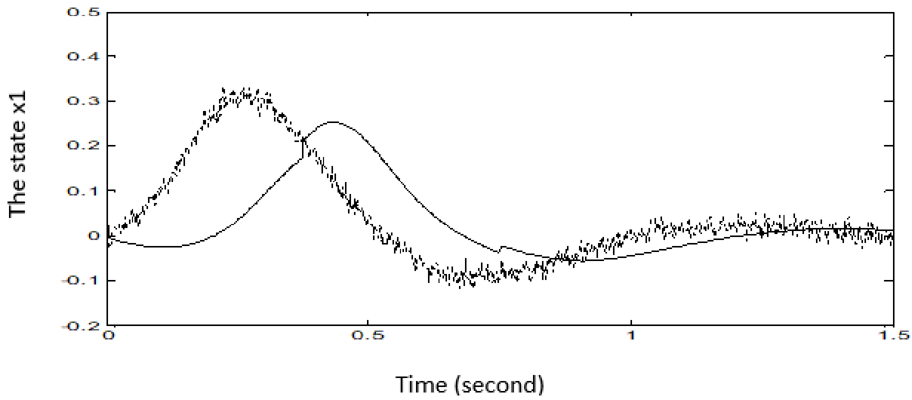

Fig. 3. A comparison between true and filtered state of x_1.

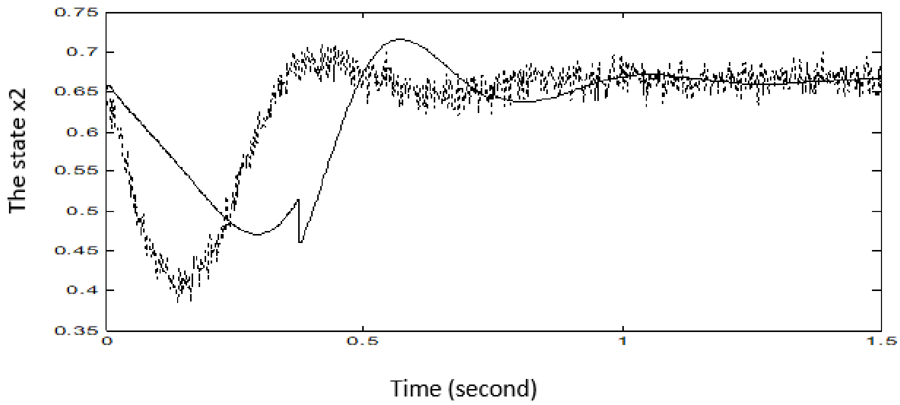

Fig. 4. A comparison between true and filtered state of x_2.

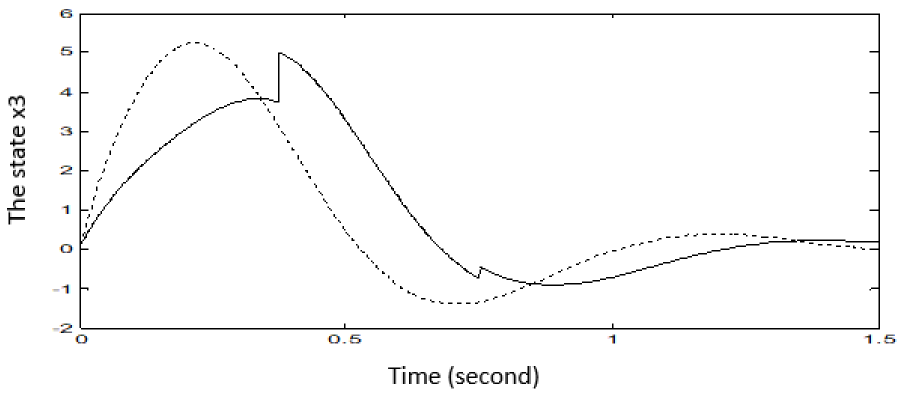

Fig. 5. A comparison between true and filtered state of x_3.

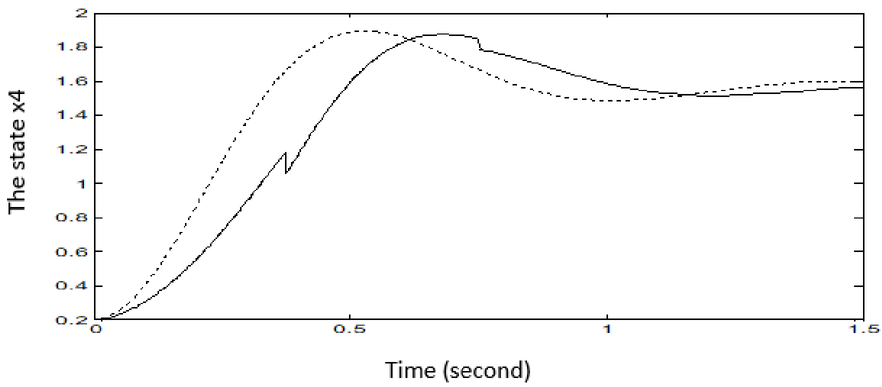

Fig. 6. A comparison between true and filtered state of x_4.

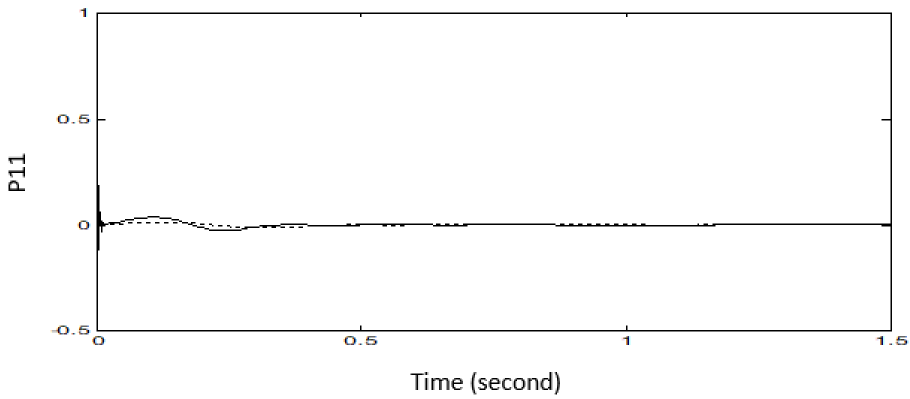

Fig. 7. A comparison between conditional variance trajectories.

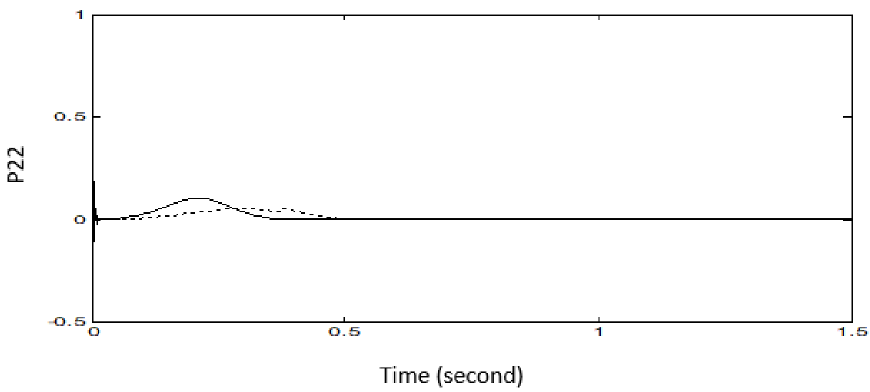

Fig. 8. A comparison between conditional variance trajectories.

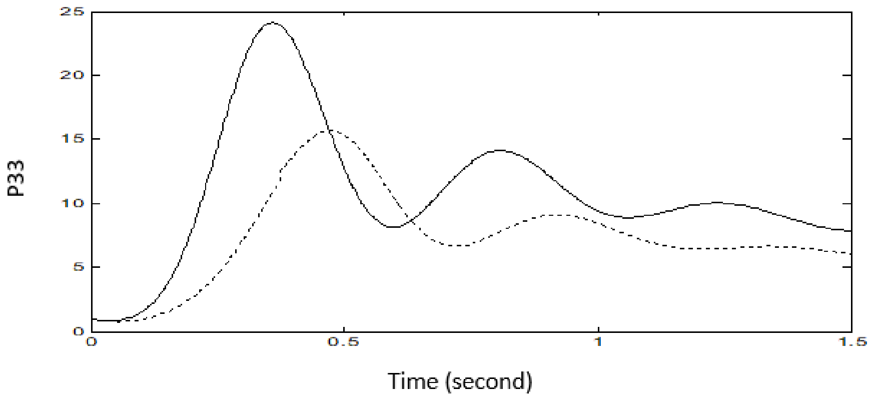

Fig. 9. A comparison between conditional variance trajectories.

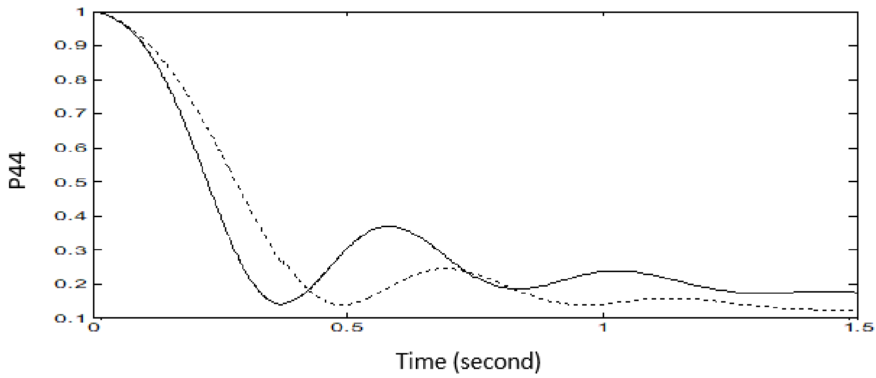

Fig. 10. A comparison between conditional variance trajectories.

The second set of system parameters, initial conditions are the same as the first set but at larger measurement noise variance, namely $\eta = 5$.

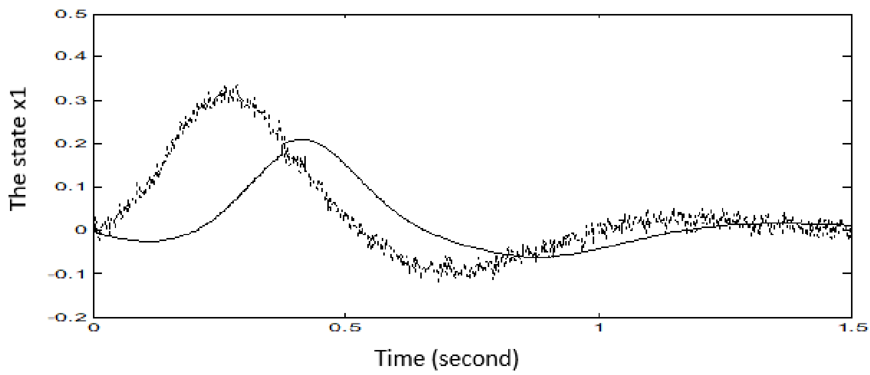

Fig. 11. A comparison between true and filtered state of x_1.

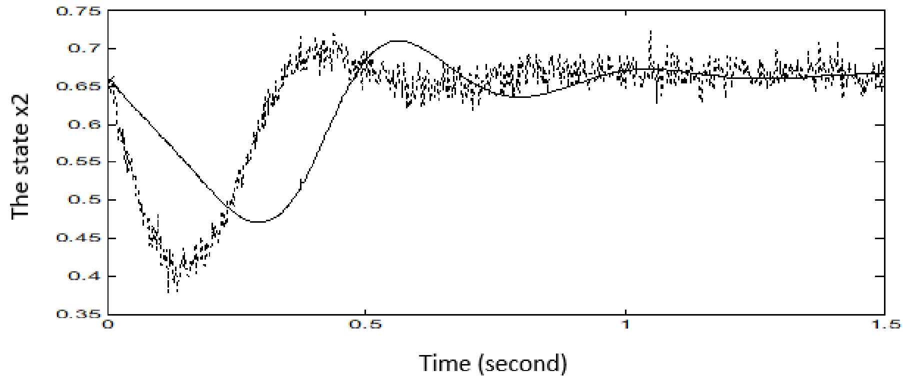

Fig. 12. A comparison between true and filtered state of x_2.

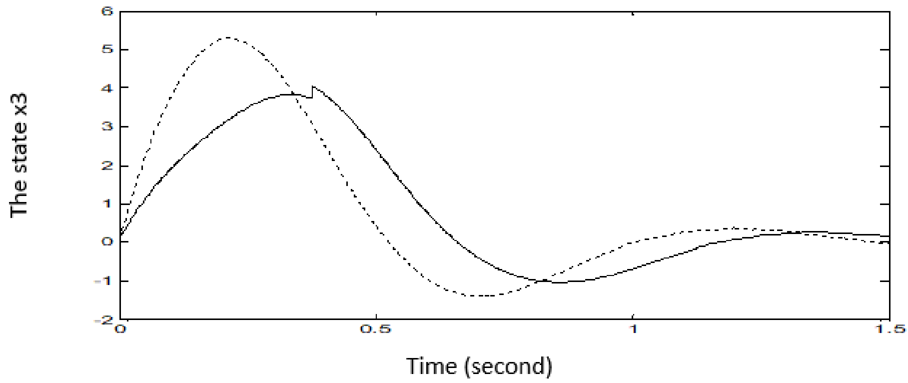

Fig. 13. A comparison between true and filtered state of x_3.

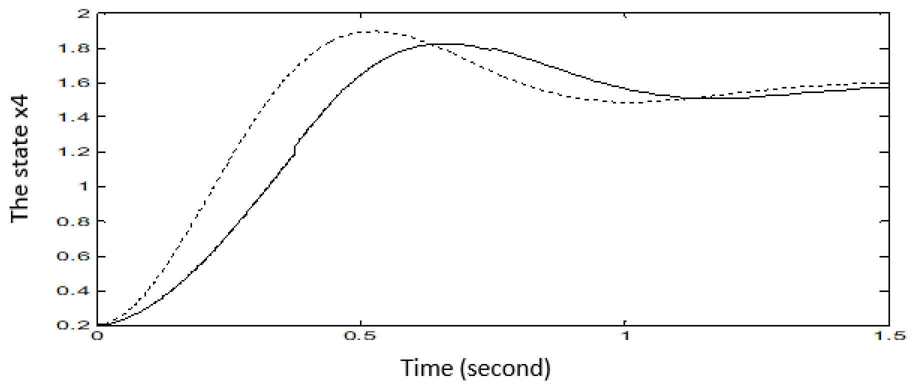

Fig. 14. A comparison between true and filtered state of x_4.

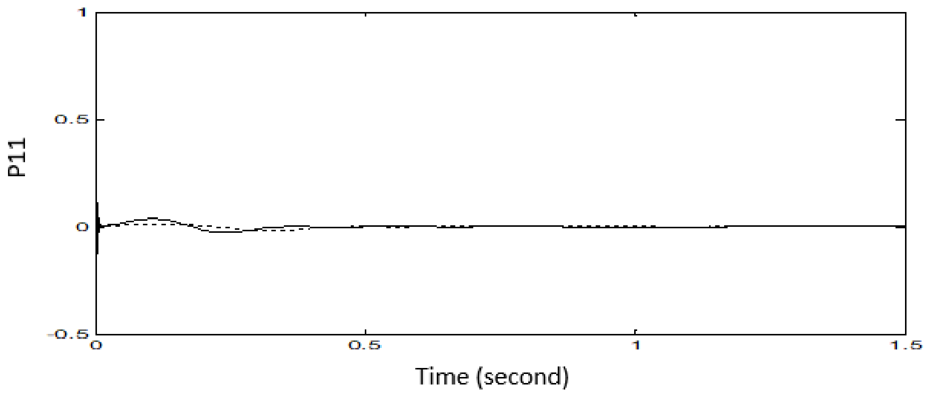

Fig. 15. A comparison between conditional variance trajectories.

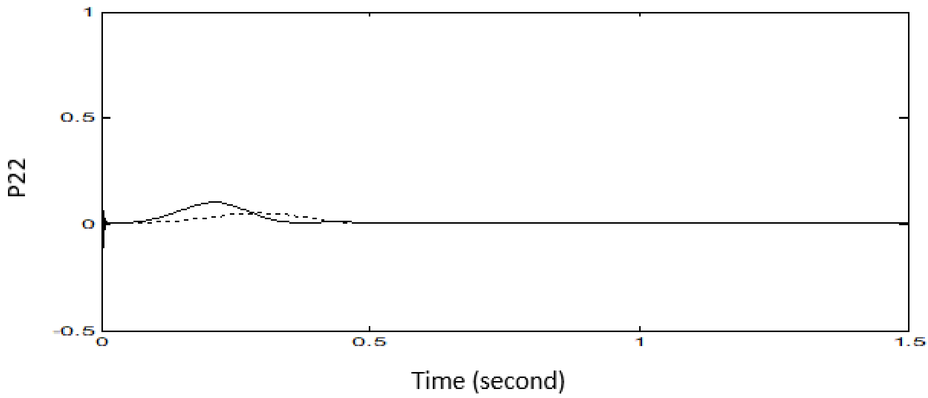

Fig. 16. A comparison between conditional variance trajectories.

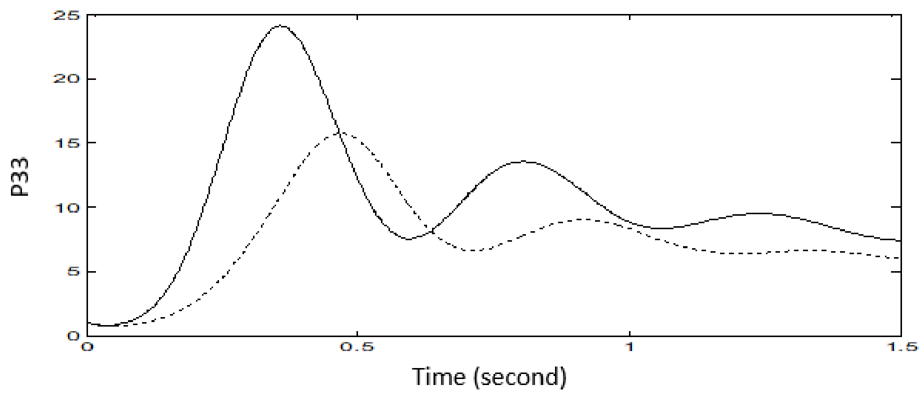

Fig. 17. A comparison between conditional variance trajectories.

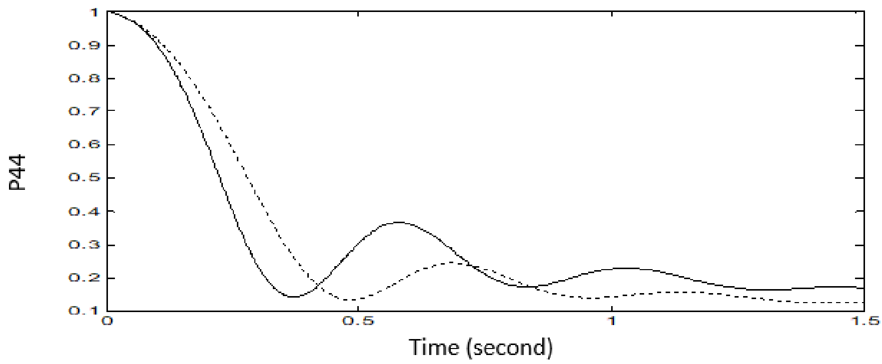

Fig. 18. A comparison between conditional variance trajectories.

Figures 11 to 14 give numerical simulation results of the true and filtered trajectories of the states of the two-phase PMSM dynamics. The dotted line denotes the true trajectories of all states. The solid line denotes higher-order 'filtered' state trajectories. The extended Kalman filtering trajectories are displayed via the solid line and higher-order filtering equations are demonstrated in Figures 15 to 18 via the dotted line.

The numerical simulation results given in Figures 7 to 10 and 11 to 14 suggest the superiority of non-linear filtering equations (17)-(32) in comparison with EKF equations (37)-(52). This paper recommends the non-filtering equations stated in (17)-(32) for analysing the stochasticity of the two-phase PMSM dynamics in the conditional variance sense. Figures 7 to 10 and Figures 15 to 18 illustrate the qualitative characteristics of the proposed filter in the conditional variance sense that suggests relatively less variances of the states $(x_1, x_2, x_3, x_4)^T$ of the two-phase PMSM dynamic in lieu of the extended Kalman filter. The reduced variances are attributed to additional correction terms, the terms $\frac{1}{2}\sum_{p,q} P_{pq}\partial_{\langle x_p\rangle\langle x_q\rangle} a_i(\langle x_\tau\rangle, \tau)$, $\frac{1}{2}\sum_{p,q} P_{pq}\partial_{\langle x_p\rangle\langle x_q\rangle}(bb^T)_{ij}(\langle x_\tau\rangle, \tau)$ of in between the observation equations (2), (3) associated with variance evolutions of the proposed filter which were absent in EKF equations. As state independent process noise acting in the system dynamics no effect of term $\frac{1}{2}\sum_{p,q} P_{pq}\partial_{\langle x_p\rangle\langle x_q\rangle}(bb^T)_{ij}(\langle x_\tau\rangle, \tau)$ reflected on estimation equations. The terms $\frac{1}{2}\sum_{p,q,r,s} P_{pr}P_{qs}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle\left\langle x_q^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k) \times \partial_{\left\langle x_r^{t_{k-1}}\right\rangle\left\langle x_s^{t_{k-1}}\right\rangle} h^T(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)$ and $\frac{1}{2}\sum_{p,q} P_{pq}\partial_{\left\langle x_p^{t_{k-1}}\right\rangle\left\langle x_q^{t_{k-1}}\right\rangle} h(\left\langle x_{t_k}^{t_{k-1}}\right\rangle, t_k)$ of at the observation equations (5) and (6), which were absent in EKF equations but due to linear observations no effect reflected on the results of this paper.

4. Conclusion

This paper developed a continuous-discrete filter for a stochastic two-phase PMSM dynamics. The continuous-discrete filtering equations (17) to (30) will be greatly useful to develop the stochastic adaptive control law' of the two-phase PMSM dynamics as well, where observations are available at the discrete-time instants. The structure of filtering equations of the paper explains the contributions of the predicted estimate and observation noise 'explicitly' to the filtered estimate. More precisely, equations (3) to (6) and (17) to (30) are the main analytic results of this paper.

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