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Cohomology Theory for Digital Images

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Abstract. In this paper we propose a mathematical framework that can be used for defining cohomology of digital images. We state the Eilenberg-Steenrod axioms and the Universal Coefficient Theorem for this cohomology theory. We show that the Künneth formula doesn't hold. A cup product is defined and its main properties are proved.

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1. Introduction

Homology is an algebraic invariant and cohomology is a functional algebraic variant of homology. In terms of main information there is not a big difference between homology and cohomology. The cohomology satisfies the Eilenberg-Steenrod axioms much like the axioms for homology (see [15]). Homology has in general a $A(\infty)$-coalgebra structure. The main difference between homology and cohomology is that cohomology groups are contravariant functors while homology groups are covariant. As groups cohomology does not give anything that homology does not already provide. Whatever geometric interpretation you have for homology would mostly probably work also for cohomology. The homology groups of a space determine its cohomology groups, and the converse holds at least when the homology groups are finitely generated. But the multiplication in cohomology, called the cup product, which makes the cohomology groups of a space into a ring. The cup product allows better differentiation between topological spaces which is not possible with homology. In this sense cohomology is a finer invariant. A central advantage of cohomology theory over homology, at least in terms of structure and strength as an invariant, is the additional ring structure from the cup product. There are spaces with the same homology and cohomology as groups, but where the ring structure on the cohomologies is different; in this case one can use the cup product to distinguish them.

Topological invariants are extremely useful in many applications related to digital imaging and geometric modeling. Computing topological invariants of objects has a significant impact in digital images. The fundamental group is a homotopical invariant (combinatorial, not algebraic) and the previous assertion is really diffuse and it is not true for digital objects of dimension higher than two. It has been studied by many researchers [3, 4, 6, 7, 9, 22].

During the last years, important applications in science and engineering are done by the homology and cohomology theories. Problems in geometric modelling, digital image processing, dynamical systems and material science activate the progress of computational approach to homology and cohomology theories (see [18]). The development in computer science, however, enables that progress. Algebraic topology and its applications waited until the modern creation of powerful computers due to a very high complexity of operations, despite algebraic topology has arised from applications at its early time. Cohomology theory is more difficult than homology but not less important in terms of applications.

Fundamental properties of digital simplicial cohomology groups are not so widely discussed in the literature so one contribution of this work is to report some recent works and bring these results to the imagery community.

Gonzalez-Diaz and Real [10] propose a method for computing the cohomology ring of 3D digital binary valued pictures through a simplicial complex topologically representing the picture. Diaz-Pernil et al. (see [9]) present a new solution for the homology groups of binary 2D image problem by using membrane computing techniques. This problem tries to calculate the number of connected components from a given binary 2D image. Gonzalez-Diaz et al. [12] give formulas to compute the cohomology ring of 3D cubical complexes and develop a method for the computation on 3D binary-valued pictures. Gonzalez-Diaz et al. [11] study cohomology in the structural pattern recognition and introduce an algorithm to efficiently compute representative cocycles which is basic elements of cohomology in 2D.

Kaczynski and Mrozek develop a cohomology ring algorithm in a dimension independent framework of combinatorial cubical complexes (see [18]) with the aim of applying it to the topological analysis of high-dimensional data.

Berciano et al. [2] give a direct computational application of Homological Perturbation Theory to computer imagery. This computational tool is useful for distinguishing 3D digital images. Molina-Abril and Real [23] introduce a 2D homology-based digital image processing framework.

Karaca and Ege [19] construct the digital cubical homology groups of digital images and study some its fundamental properties. They study some results about the simplicial homology of 2D digital images (see [20]). They [21] give characteristic properties of the simplicial homology groups of digital images and investigate Eilenberg-Steenrod axioms for the simplicial homology groups of digital images.

Pilarczyk and Real [25] introduce algorithms to compute homology, cohomology and related operations on cubical cell complexes, using a technique based on a chain contraction from the original chain complex to a reduced one that represents its homology.

This paper is organized as follows: Section 2 provides the general notions of digital images with κ-adjacency relations, definitions and theorems related to digital homotopy, digital fundamental group. In Section 3 it is given some definitions and some fundamental properties of digital simplicial cohomology groups are given. In Section 4 we express the Universal Coefficient Theorem for digital images and give an example about this theorem. In Section 5 we show that the Künneth formula for simplicial cohomology doesn't hold for digital images. In the last Section we finally define the cup product and prove its basic properties.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers. *A (binary) digital image* is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ represents certain adjacency relation for the members of X. There are various adjacency relations in the study of digital images, we give one of them. Two points p and q in $\mathbb{Z}^2$ are 8-*adjacent* if they are distinct and differ by at most 1 in each coordinate; points p and q in $\mathbb{Z}^2$ are 4-*adjacent* if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are 26-*adjacent* if they are distinct and differ by at most 1 in each coordinate; they are 18-*adjacent* if they are 26-adjacent and differ in at most two coordinate; they are 6-*adjacent* if they are 18-adjacent and differ in exactly one coordinate.

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. *A κ-neighbor* of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is *κ-connected* [16] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r-1$.

Let $a, b \in \mathbb{Z}$ with $a < b$. *A digital interval* [3] is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}.$$

Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \longrightarrow Y$ is said to be (κ_0, κ_1)-*continuous* [4, 26] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y.

A $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$ is called *a digital κ-path* [5] from x to y in a digital image X. *A simple closed κ-curve* of $m \geq 4$ points in a digital image X is a sequence $\{f(0), f(1), \ldots, f(m-1)\}$ of images of the κ-path $f : [0, m-1]_{\mathbb{Z}} \longrightarrow X$ such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm \mod m$.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. A function $f : X \longrightarrow Y$ is (κ_0, κ_1)-*isomorphism* [7] if f is (κ_0, κ_1)-continuous and bijective and also $f^{-1} : Y \longrightarrow X$ is (κ_1, κ_0)-continuous.

Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. Two (κ_0, κ_1)-continuous functions $f, g : X \longrightarrow Y$ are said to be *digitally* (κ_0, κ_1)-*homotopic* in Y [4] if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \longrightarrow Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$; for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \longrightarrow Y$ defined by

$$H_x(t) = H(x, t) \quad \text{for all } t \in [0, m]_{\mathbb{Z}},$$

is $(2, \kappa_1)$-continuous; and for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \longrightarrow Y$ defined by

$$H_t(x) = H(x, t) \quad \text{for all } x \in X,$$

is (κ_0, κ_1)-continuous. The function H is called *a digital* (κ_0, κ_1)-*homotopy* between f and g. A digital image (X, κ) is said to be κ-*contractible* [3] if its identity map is (κ, κ)-homotopic to a constant function $\bar{c}$ for some $c \in X$ where the constant function $\bar{c} : X \longrightarrow X$ is defined by $\bar{c}(x) = c$ for all $x \in X$.

Let $f : X \longrightarrow Y$ be a (κ, λ)-continuous function and let $g : Y \longrightarrow X$ be a (λ, κ)-continuous function such that

$$f \circ g \simeq_{\lambda,\lambda} 1_X \quad \text{and} \quad g \circ f \simeq_{\kappa,\kappa} 1_Y.$$

Then we say X and Y have the same (κ, λ)-*homotopy type* [5] and that X and Y are (κ, λ)-*homotopy equivalent.*

For a digital image (X, κ) and its subset (A, κ), we call (X, A) a digital image pair with κ-adjacency. Moreover, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital image.

Let S be a set of nonempty subset of a digital image (X, κ). Then the members of S are called *simplexes* of (X, κ) if the following hold [29] :

- If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.
- If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

An m-simplex is a simplex S such that $|S| = m+1$. Let P be a digital m-simplex. If $P^{'}$ is a nonempty proper subset of P, then $P^{'}$ is called *a face* of P. We write $Vert(P)$ to denote the vertex set of P, namely, the set of all digital 0-simplexes in P. A digital subcomplex A of a digital simplicial complex X with κ-adjacency is a digital simplicial complex contained in X with $Vert(A) \subset Vert(X)$.

Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. If the following statements hold then (X, κ) is called *a finite digital simplicial complex* [1] :

- If P belongs to X, then every face of P also belongs to X.
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

We give some examples about digital simplices in the following:

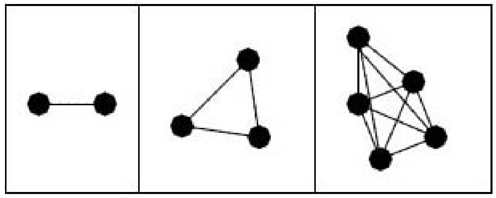

Fig. 1. (2,1), (8,2) and (26,4)-simplices.

Let $\varphi : (X, \kappa_0) \longrightarrow (Y, \kappa_1)$ be a function between digital images. φ is called *a digital simplicial map* [8] if for every digital (κ_0, m)-simplex P determined by the adjacency relation κ_0 in X, $\varphi(P)$ is a (κ_1, n)-simplex in Y for some $n \le m$.

The class of all digital simplicial complexes together with all digital simplicial maps between digital simplicial complexes, where composition is the usual function composition, forms a category.

The dimension of a digital simplicial complex X is the largest integer m such that X has an m-simplex. $C_q^\kappa(X)$ is a free abelian group [1] with basis all digital (κ, q)-simplices in X. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C_q^\kappa(X)$ is a trivial group. The homomorphism $\partial_q : C_q^\kappa(X) \longrightarrow C_{q-1}^\kappa(X)$ defined (see [1]) by

$$\partial_q(< p_0, p_1, \ldots, p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, \ldots, \hat{p_i}, \ldots, p_q >, & q \le m \\ 0, & q > m \end{cases}$$

is called a boundary homomorphism, where $\hat{p_i}$ means delete the point p_i.

Example 2.1. Let $X = [0,1]_{\mathbb{Z}}$ be a digital image with 2-adjacency. We can direct X by the ordering $0 < 1$. $C_0^2(X)$ and $C_1^2(X)$ are free abelian groups with bases

$$\{< 0 >, < 1 >\} \quad \text{and} \quad \{< 0, 1 >\}$$

respectively. So we have a short following sequence:

$$0 \xrightarrow{\partial_2} C_1^2(X) \xrightarrow{\partial_1} C_0^2(X) \xrightarrow{\partial_0} 0$$

By the definition of homomorphism, we have

$$\partial_1(a < 0, 1 >) = a(< 1 > - < 0 >),$$

where $a \in \mathbb{Z}$.

Corollary 2.2 [1]. For all $1 \le q \le m$, we have $\partial_{q-1} \circ \partial_q = 0$.

In [1], it is shown that the sequence

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} C_{m-1}^\kappa(X) \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a digital chain complex.

Definition 2.3 [1]. Let (X, κ) be a digital simplicial complex.

- $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called *the group of digital simplicial q-cycles.*
- $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called *the group of digital simplicial q-boundaries.*
- $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ is called *the qth digital simplicial homology group.*

3. Digital Simplicial Cohomology Groups

We define the cohomology groups of a digital simplicial complex.

Definition 3.1 [24]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex. The *simplicial digital cochain complex* $(\mathcal{C}^*(X), \delta)$ is defined as follow. For any $q \in \mathbb{Z}$, the *q-dimensional digital cochain group* is

$$C^{q,\kappa}(X) = Hom(C_q^\kappa(X), \mathbb{Z}),$$

where Hom is the functor assigning to any abelian group G the group of all homomorphisms from G to $\mathbb{Z}$, called the dual of G. Elements of $C^{q,\kappa}(X)$ are called digital cochains and denoted either by c^q or by c^*, if we don't need to specify their dimension q. The value of a digital cochain c^q on a chain d^q is denoted by $< c^q, d^q >$. The qth coboundary map $\delta^k : C^{q,\kappa}(X) \longrightarrow C^{q+1,\kappa}(X)$ is the dual homomorphism of ∂_{q+1} defined by

$$< \delta^q c^q, d_{q+1} > := < c^q, \partial_{q+1} d_{q+1} > .$$

Note that $C^{q,\kappa}(X)$ is the free abelian group generated by the dual canonical basis $\{Q^* \mid Q \in C_q^\kappa(X)\}$ where

$$< Q^*, P > = \begin{cases} 1, & P = Q \\ 0, & P \neq Q. \end{cases}$$

Definition 3.2 [24]. Given a digital simplicial complex (X, κ), the *group of q-dimensional cocycles of* (X, κ) is $Z^{q,\kappa}(X) := \text{Ker } \delta^q$, and *the group of q-dimensional coboundaries of* (X, κ) is $B^{q,\kappa} := \text{Im } \delta^{q-1}$. The qth *simplicial cohomology group of* (X, κ) is

$$H^{q,\kappa}(X) := Z^{q,\kappa}(X)/B^{q,\kappa}(X).$$

Right now, we give definition of a relative simplicial homology for digital images. Let (A, κ) be a digital subcomplex of the digital simplicial complex (X, κ). The cochain group $C^{q,\kappa}(A)$ is a subgroup of the cochain group $C^{q,\kappa}(X)$. The quotient group $C^{q,\kappa}(X)/C^{q,\kappa}(A)$ is called the group of relative chains of X modulo A and is denoted by $C^{q,\kappa}(X, A)$. The boundary operator

$$\delta^q : C^{q,\kappa}(A) \longrightarrow C^{q+1,\kappa}(A)$$

is the restriction of the boundary operator on $C^{q,\kappa}(X)$. A homomorphism

$$C^{q,\kappa}(X,A) \longrightarrow C^{q+1,\kappa}(X,A)$$

of the relative chain groups is induced by δ^q and is also denoted by δ^q.

Definition 3.3 [24]. Let (A,κ) be a digital subcomplex of the digital simplicial complex (X,κ).

- $Z^{q,\kappa}(X,A) = \text{Ker } \delta^q$ is called *the group of digital relative simplicial q-cocycles.*
- $B^{q,\kappa}(X,A) = \text{Im } \delta^{q-1}$ is called *the group of digital relative simplicial q-coboundaries.*
- $H^{q,\kappa}(X,A) = Z^{q,\kappa}(X,A)/B^{q,\kappa}(X,A)$ is called *the qth digital relative simplicial cohomology group.*

Theorem 3.4. *For each $q \geq 0$, $H^{q,\kappa}$ is a contravariant functor from the category of digital simplicial complexes and simplicial maps to the category of abelian groups.*

Proof. If $\phi : (X,\kappa_0) \longrightarrow (Y,\kappa_1)$ is a digital simplicial map, define

$$\phi^* : H^{q,\kappa_0}(X) \longrightarrow H^{q,\kappa_1}(Y)$$

by $\phi^*(z + B^{q,\kappa_0}(X)) = \phi^*(z) + B^{q,\kappa_1}(Y)$, where $z \in Z^{q,\kappa_0}(X)$. Let

$$1_{(X,\kappa_0)} : (X,\kappa_0) \longrightarrow (X,\kappa_0)$$

be the identity. Then

$$\begin{aligned}
[1_{(X,\kappa_0)}]^*(z + B^{q,\kappa_0}(X)) &= (1_{(X,\kappa_0)})^*(z) + B^{q,\kappa_0}(X) \\
&= (1_{(X,\kappa_0)} \circ z) + B^{q,\kappa_0}(X) \\
&= z + B^{q,\kappa_0}(X) \\
&= 1_{H^{q,\kappa_0}(X)}(z + B^{q,\kappa_0}(X))
\end{aligned}$$

that is, $[1_{(X,\kappa_0)}]^* = 1_{H^{q,\kappa_0}(X)}$. Let

$$\phi : (X,\kappa_0) \longrightarrow (Y,\kappa_1) \quad \text{and} \quad \psi : (Z,\kappa_2) \longrightarrow (X,\kappa_0).$$

Then

$$\begin{aligned}
(\phi \circ \psi)^*(z + B^{q,\kappa_2}(Z)) &= ((\phi \circ \psi) \circ z) + B^{q,\kappa_1}(Y) \\
&= (\phi \circ (\psi \circ z) + B^{q,\kappa_1}(Y) \\
&= \phi^*((\psi \circ z) + B^{q,\kappa_0}(X)) \\
&= \phi^*(\psi^*(z + B^{q,\kappa_2}(Z))) \\
&= \phi^* \circ \psi^*(z + B^{q,\kappa_2}(Z))
\end{aligned}$$

where $z \in B_q^{\kappa_2}(Z)$. So $(\phi \circ \psi)^* = \phi^* \circ \psi^*$. □

Example 3.5. Let (X, κ) be a single vertex. Havana et.al. [1] have shown that

$$H_q^{\kappa}(X) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

Since for $q > 0$,

$$C^{q,\kappa}(X) = Hom(C_q^{\kappa}(X), \mathbb{Z}) = 0,$$

we get $H^{q,\kappa}(X) = 0$. On the other hand, for $q = 0$

$$C^{0,\kappa}(X) = Hom(C_0^{\kappa}(X), \mathbb{Z}) = \mathbb{Z}.$$

So there is a short sequence :

$$0 \xrightarrow{\delta^0} C^{0,\kappa}(X) \xrightarrow{\delta^1} 0$$

Since Ker $\delta^1 = \mathbb{Z}$ and Im $\delta^0 = 0$, we find that $H^{0,\kappa}(X) = \mathbb{Z}$.

Example 3.6. Let $X = [0,1]_{\mathbb{Z}}$ be a digital image with 2-adjacency. Since X is 2-contractible digital image, from Example 3.5 we conclude that

$$H^{q,2}(X) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

Example 3.7. Let X be a digital simple closed κ-curve. In [1], it is shown that

$$H^{q,\kappa}(X) = \begin{cases} \mathbb{Z}, & q = 0, 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Now we state the Eilenberg-Steenrod axioms for the digital simplicial cohomology. We don't give proofs of these axioms because they are similar as in algebraic topology.

Axiom 1 (Identity axiom) [24]. Let X be a digital image with κ-adjacency. If $i : (X, \kappa) \longrightarrow (X, \kappa)$ is the identity, then the induced homomorphism $i^* : H^{*,\kappa}(X) \longrightarrow H^{*,\kappa}(X)$ is the identity.

Axiom 2 (Dimension axiom) [24]. If (X, κ) is a one-point digital image, then

$$H^{q,\kappa}(X, G) = \begin{cases} G, & q = 0 \\ 0, & q > 0. \end{cases}$$

Axiom 3 (Composition axiom) [24]. Let X, Y and Z be digital images with κ_0, κ_1 and κ_2-adjacency, respectively. If $f : (X, \kappa_0) \longrightarrow (Y, \kappa_1)$ and $g : (Y, \kappa_1) \longrightarrow (Z, \kappa_2)$ are digitally continuous functions, then $(f \circ g)^* = f^* \circ g^*$.

Axiom 4 (Commutativity axiom) [24]. Let (X, A) and (Y, B) be digital image pairs with κ_0 and κ_1-adjacency, respectively. If $f : (X, A) \longrightarrow (Y, B)$, then the following diagram commutes :

$$\begin{array}{ccc} H^{q-1,\kappa_1}(Y,B) & \xrightarrow{f^*} & H^{q-1,\kappa_0}(X,A) \\ \delta^* \downarrow & & \downarrow \delta^* \\ H^{q,\kappa_1}(B) & \xrightarrow{(f|_A)^*} & H^{q,\kappa_0}(A) \end{array}$$

Axiom 5 (Exactness axiom) [24]. The long sequence

$$\ldots \longrightarrow H^{q,\kappa}(X,A) \xrightarrow{p^*} H^{q,\kappa}(X) \xrightarrow{i^*} H^{q,\kappa}(A) \xrightarrow{\delta^*} H^{q+1,\kappa}(X,A) \longrightarrow \ldots$$

is exact, where $i : A \longrightarrow X$ and $p : X \longrightarrow (X, A)$ are inclusion maps.

Definition 3.8 [13]. Let $c^* := (x_0, x_1, \ldots, x_n)$ be a closed κ-curve in $\mathbb{Z}^2$, where $\kappa, \bar{\kappa} = 4, 8$. A point x of the complement $\bar{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be *interior* to c^* if it belongs to the bounded $\bar{\kappa}$-connected component of $\bar{c^*}$. The set of all interior points to c^* is denoted by $Int(c^*)$. If (X, κ) is a digital image in $\mathbb{Z}^2$, then *interior* of X *i.e.* $Int(X)$ could be defined as $Int(X) = X$.

Since a digital image consists of discrete points, *a closure* of a digital image is again itself, that is, if A is a digital image, then its closure which is denoted by $\bar{A}$ is $\bar{A} = A$.

We have the following result:

Theorem 3.9. *The excision axiom for simplicial cohomology doesn't hold for digital images.*

Proof. Let $X = \{c_0 = (0, -1), c_1 = (1, 0), c_2 = (0, 1), c_3 = (-1, 0)\} \subset \mathbb{Z}^2$ be a digital image with 8-adjacency (see Fig. 2).

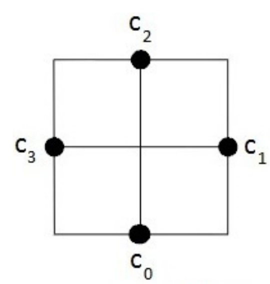

Figure 2. X

By Example 3.7, we get

$$H^{q,8}(X) = \begin{cases} \mathbb{Z}, & q = 0, 1 \\ 0, & q \neq 0, 1. \end{cases}$$

Let $A = \{c_1 = (1, 0), c_2 = (0, 1), c_3 = (-1, 0)\}$ and $U = \{c_1 = (1, 0)\}$ (see Fig. 3).

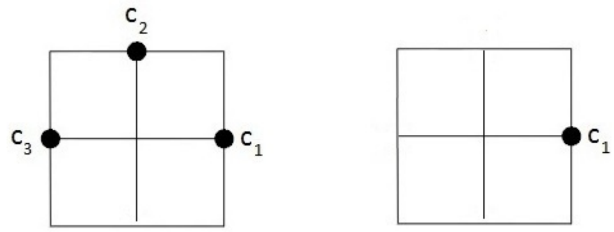

Figure 3. A and U

It's clear that $A \subset X$, $U \subset A$ and $\bar{U} \subset IntA$. We would like to compute $H^{1,8}(X-U, A-U)$ and $H^{1,8}(X, A)$. Since A is digitally 8-contractible, the digital simplicial cohomology groups of A and U are

$$H^{q,8}(A) = H^{q,8}(U) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

Using Axiom 5, we get the exact sequence :

$$\ldots \longrightarrow H^{q,8}(X, A) \longrightarrow H^{q,8}(X) \longrightarrow H^{q,8}(A) \longrightarrow H^{q+1,8}(X, A) \longrightarrow \ldots$$

For $q > 1$, as $H^{q,8}(A) = 0$ we get $H^{q,8}(X) \cong H^{q,8}(X, A)$. For $q = 0, 1$ we have the following exact sequence,

$$0 \xrightarrow{\delta^0} H^{0,8}(X, A) \xrightarrow{p^*} \mathbb{Z} \xrightarrow{i^*} \mathbb{Z} \xrightarrow{\delta^1} H^{1,8}(X, A) \xrightarrow{q^*} \mathbb{Z} \xrightarrow{j^*} 0,$$

where p, i, j are inclusion maps. Using this exact sequence, we have

$$\text{Im } q^* = \text{Ker } j^* = \mathbb{Z}$$

thus q^* is an epimorphism.

$$i^* : \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$c \longmapsto i^*(c) = c$$

Im $i^* \cong \mathbb{Z}$. From exactness of sequence, we get Ker $\delta^1 \cong \mathbb{Z}$. Then Im $\delta^1 = 0$. By the First Isomorphism Theorem, we have

$$H^{1,8}(X, A)/\text{Ker } q^* \cong \text{Im } q^*$$

and hence $H^{1,8}(X, A) = \mathbb{Z}$.

Now we are ready to compute $H^{q,8}(X - U, A - U)$. Let

$X - U = \{c_0 = (0, -1), c_2 = (0, 1), c_3 = (-1, 0)\}$ and $A - U = \{c_2 = (0, 1), c_3 = (-1, 0)\}$

(see Fig. 4).

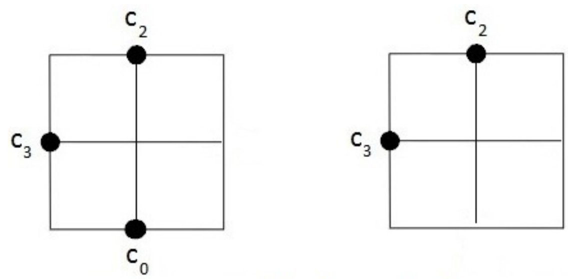

Figure 4. $X - U$ and $A - U$

It's clear that $A - U \subset X - U$. By reason of the fact that $X - U$ and $A - U$ are digitally 8-contractible, we get

$$H^{q,8}(X-U) = H^{q,8}(A-U) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

So we have the exact sequence :

$$\ldots \to H^{q,8}(X-U, A-U) \to H^{q,8}(X-U) \to H^{q,8}(A-U) \to H^{q+1,8}(X-U, A-U) \to \ldots$$

For $q > 1$, as $H^{q,8}(A-U) = 0$ we get $H^{q,8}(X-U) \cong H^{q,8}(X-U, A-U)$. For $q = 0, 1$ we have following exact sequence :

$$0 \xrightarrow{\alpha} H^{1,8}(X-U, A-U) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\delta} H^{0,8}(X-U, A-U) \xrightarrow{\lambda} 0$$

Since γ is an inclusion map, Ker $\gamma = 0$. Im $\beta = 0$ due to exactness. Using the First Isomorphism Theorem, we get

$$H^{1,8}(X-U, A-U)/\text{Ker } \beta \cong \text{Im } \beta$$

and we find that $H^{1,8}(X-U, A-U) = \text{Ker } \beta$. As this sequence is exact, we have Ker $\beta = \text{Im } \alpha = 0$ and so we get $H^{1,8}(X, A) = 0$. As a result for $q = 1$, since

$$H^{q,8}(X-U, A-U) \ncong H^{q,8}(X, A),$$

we see that the excision theorem for the simplicial cohomology doesn't hold for digital images. □

Since the simple closed curve X in Example 3.7 is 8-contractible (see [4]), it has the same $(8, 8)$-homotopy type as a single point image. By Example 3.5, $H^{1,8}(X) \cong 0$. But from Example 3.7 we have $H^{1,8}(X) \cong \mathbb{Z}$. Although X and a single point image are $(8, 8)$-homotopy equivalent, they don't have isomorphic cohomology groups. As a result, the homotopy axiom for simplicial cohomology doesn't hold for digital images.

Corollary 3.10. Two digital images that are contractible or that have the same homotopy type couldn't have isomorphic cohomology groups.

4. The Universal Coefficient Theorem for the Digital Simplicial Cohomology

We recall *Ext* functor and its basic properties (see [28]). A sequence of two homomorphisms of abelian groups $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $Im\ f = Ker\ g$. A free abelian group is an abelian group which has a basis in that every element of the group could be written in one way as a finite linear combination of elements of the basis with integer coefficients. For each abelian group A, choose an exact sequence $0 \longrightarrow R \xrightarrow{i} F \longrightarrow A \longrightarrow 0$ with F free abelian and i inclusion map. For any abelian group G, if we apply the contravariant functor $Hom(-, G)$ to this exact sequence, then we get

$$0 \longrightarrow Hom(A, G) \longrightarrow Hom(F, G) \xrightarrow{i^{\#}} Hom(R, G) \longrightarrow A \longrightarrow 0.$$

Thus we can define $Ext(A, G)$ as

$$Ext(A, G) = coker i^{\#} = Hom(R, G)/i^{\#}(Hom(F, G)).$$

Some properties of *Ext* functor are the following :
1. If E is free abelian, then $Ext(E, G) = 0$.
2. If D is divisible, then $Ext(A, D) = 0$.
3. $Ext(\sum A_i, G) \cong \prod Ext(A_j, G)$.
4. $Ext(A, \prod G_j) \cong \prod Ext(A, G_j)$.
5. $Ext(\mathbb{Z}/m\mathbb{Z}, G) = G/mG$.

We state the universal coefficient theorem for cohomology of a digital simplicial complex. We don't give its proof because it is similar as in algebraic topology.

Theorem 4.1 [26]. *Let (X, κ) be a digital simplicial complex. For any abelian group G, there is exact sequence*

$$0 \longrightarrow Ext(H_{q-1}^{\kappa}(X), G) \longrightarrow H^{q,\kappa}(X, G) \longrightarrow Hom(H_q^{\kappa}(X), G) \longrightarrow 0,$$

where $H_q^{\kappa}(X) = H_q^{\kappa}(X; \mathbb{Z})$. This exact sequence is split; hence

$$H^{q,\kappa}(X, G) \cong Hom(H_q^{\kappa}(X), G) \oplus Ext(H_{q-1}^{\kappa}(X), G).$$

Example 4.2. Let

$$MSS_{18}' = \{c_0 = (1,1,0), c_1 = (0,2,0), c_2 = (-1,1,0), c_3 = (0,0,0), c_4 = (0,1,-1), c_5 = (0,1,1)\} \subset \mathbb{Z}^3$$

be a digital image with 18-adjacency (see Fig. 5).

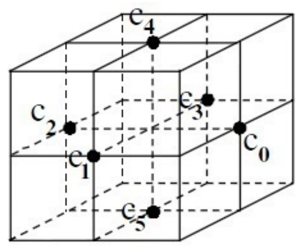

Figure 5. [14] MSS'_{18}

In [8], it is shown that digital simplicial homology groups of MSS'_{18} are

$$H_q^{18}(MSS'_{18}) = \begin{cases} \mathbb{Z}, & q = 0,2 \\ 0, & q \neq 0,2. \end{cases}$$

We use the Universal Coefficient Theorem for cohomology to calculate simplicial cohomology groups of MSS'_{18} with $\mathbb{Z}_2$-coefficient. For $q = 0$, we compute

$$H^{0,18}(MSS'_{18};\mathbb{Z}_2) \cong Hom(\mathbb{Z};\mathbb{Z}_2) \oplus Ext(0,\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

For $q = 1$, we have

$$H^{1,18}(MSS'_{18};\mathbb{Z}_2) \cong Hom(0;\mathbb{Z}_2) \oplus Ext(\mathbb{Z},\mathbb{Z}_2) \cong 0.$$

For $q = 2$, we get

$$H^{2,18}(MSS'_{18};\mathbb{Z}_2) \cong Hom(\mathbb{Z};\mathbb{Z}_2) \oplus Ext(0,\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

For all $q > 2$, it's clear that $H^{q,18}(MSS'_{18};\mathbb{Z}_2) \cong 0$. As a result,

$$H^{q,18}(MSS'_{18};\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0,2 \\ 0, & q \neq 0,2. \end{cases}$$

5. The Künneth Formula for the Digital Simplicial Cohomology

We recall Tor functor and its basic properties [28]. The tensor product $\otimes$ of two abelian groups A and B is the abelian group having $A \times B$ as generators and

$$(a + a', b) = (a, b) + (a', b) \quad \text{and} \quad (a, b + b') = (a, b) + (a, b')$$

for all $a, a' \in A$ and all $b, b' \in B$ as relations. If F is the free abelian group with basis $A \times B$ and if N is the subgroup of F generated by all relations, then $A \otimes B = F/N$. For each abelian group A, choose an exact sequence $0 \longrightarrow R \xrightarrow{i} F \longrightarrow A \longrightarrow 0$

with F free abelian and i inclusion map. For any abelian group B, if we apply the covariant functor $-\otimes B$, then we get

$$0 \longrightarrow R\otimes B \overset{i^{\#}\otimes 1_B}{\longrightarrow} F\otimes B \longrightarrow A\otimes B \longrightarrow 0.$$

Thus $Tor(A,B)$ could be defined as

$$Tor(A,B) = Ker(i^{\#}\otimes 1_B).$$

Some properties of Tor functor are the following :
1. $Tor(\ ,B) : Ab \longrightarrow Ab$ is an additive covariant functor.
2. If D is divisible, then $Tor(A,D) = 0$.
3. $Tor(\sum A_i, B) \cong Tor(A_i, B)$ and $Tor(A, \sum B_j) \cong Tor(A, B_j)$.
4. $Tor(\mathbb{Z}/m\mathbb{Z}, B) \cong B[m] = \{b \in B : mb = 0\}$.
5. $Tor(A,B) \cong Tor(B,A)$ for all A and B.

Let us recall the Künneth formula for simplicial cohomology used in standard literature on algebraic topology.

Theorem 5.1 [24]. *Suppose X and Y are topological spaces. We then have the following relation between cohomology groups of X, Y and the product space $X\times Y$. For any $n \geq 0$ and any module M over a principal ideal domain R, we have:*

$$H^n(X\times Y; M) \cong \sum_{i+j=n} H^i(X;M)\otimes H^j(Y;M) \oplus \sum_{p+q=n-1} Tor(H^p(X;M), H^q(Y;M))$$

where Tor denotes the Tor functor. If $M = \mathbb{F}$ is a field, then the Tor functor is always trivial and in this case Künneth formula can be stated as

$$H^n(X\times Y;\mathbb{F}) \cong \sum_{i+j=n} H^i(X;\mathbb{F})\otimes H^j(Y;\mathbb{F})$$

for any $n \geq 0$.

Theorem 5.2. *The Künneth formula for simplicial cohomology doesn't hold for digital images.*

Example 5.3. Let $X = [0,1]_{\mathbb{Z}}$ and $Y = [0,1]_{\mathbb{Z}}$ be digital images with 2-adjacency. Assume that $M = \mathbb{Z}$. Then cohomology groups of X and Y are

$$H^{n,2}(X) = H^{n,2}(Y) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Since $X\times Y$ is digital simple closed 4-curve, from Example 3.7 we have

$$H^{n,4}(X\times Y) = \begin{cases} \mathbb{Z}, & n = 0,1 \\ 0, & n \neq 0,1. \end{cases}$$

We now use the Künneth formula to compute $H^{n,4}(X \times Y)$ for $n \geq 0$. Since $M = \mathbb{Z}$,

$$H^{n,4}(X \times Y; \mathbb{Z}) \cong \sum_{i+j=n} H^{i,2}(X; \mathbb{Z}) \otimes H^{j,2}(Y; \mathbb{Z}).$$

For $n = 0$, we have

$$H^{0,4}(X \times Y; \mathbb{Z}) \cong H^{0,2}(X; \mathbb{Z}) \otimes H^{0,2}(Y; \mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

For $n = 1$, we get

$$H^{1,4}(X \times Y; \mathbb{Z}) \cong (H^{0,2}(X; \mathbb{Z}) \otimes H^{1,2}(Y; \mathbb{Z})) \oplus (H^{1,2}(X; \mathbb{Z}) \otimes H^{0,2}(Y; \mathbb{Z})) \cong 0$$

For all $n \geq 2$, we have $H^{n,4}(X \times Y; \mathbb{Z}) \cong 0$. Therefore we conclude that

$$H^{n,4}(X \times Y) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Since cohomology group of $X \times Y$ for $n = 1$ is $H^{1,4}(X \times Y; \mathbb{Z}) = \mathbb{Z}$, we get a contradiction. As a result, the Künneth formula for simplicial cohomology doesn't hold for digital images.

6. The Simplicial Cup Product for Digital Images

The first goal of this section is to provide an explicit formula for computing the cup product, when (X, κ) is a digital simplicial complex.

Definition 6.1. [26]. Let (X, κ) be a digital simplicial complex. Suppose that the coefficient group G is the additive group of a commutative associative ring with identity. The *digital simplicial cup product*

$$\smile : C^{p,\kappa}(X, G) \times C^{q,\kappa}(X, G) \longrightarrow C^{p+q,\kappa}(X, G)$$

of cochains c^p and c^q is given in [15]

$$< c^p \smile c^q, [v_0, \ldots, v_{p+q}] >=< c^p, [v_0, \ldots, v_p] > . < c^q, [v_p, \ldots, v_{p+q}] >$$

where $v_0 < \ldots < v_{p+q}$ and . is the product in G.

Now we give an important properties of digital simplicial cup product. The proofs of the following theorems are analogues to algebraic topology (see [26]).

Theorem 6.2. *The digital simplicial cup product is bilinear.*

Proof. Let $\alpha, \alpha_1, \alpha_2 \in H^{p,\kappa}(X, G_1)$ and $\beta, \beta_1, \beta_2 \in H^{q,\kappa}(X, G_2)$. Since

$$< (\alpha_1 + \alpha_2) \smile \beta, [v_0, \ldots, v_{p+q}] > =< (\alpha_1 + \alpha_2), [v_0, \ldots, v_p] > . < \beta, [v_p, \ldots, v_{p+q}] >$$

$$= (< \alpha_1, [v_0, \ldots, v_p] > + < \alpha_2, [v_0, \ldots, v_p] >). < \beta, [v_p, \ldots, v_{p+q}] >$$

$=< \alpha_1, [v_0, \ldots, v_p] > . < \beta, [v_p, \ldots, v_{p+q}] > +$

$+ < \alpha_2, [v_0, \ldots, v_p] > . < \beta, [v_p, \ldots, v_{p+q}] >$

$=< \alpha_1 \smile \beta, [v_0, \ldots, v_{p+q}] > + < \alpha_2 \smile \beta, [v_0, \ldots, v_{p+q}] >$

$=< \alpha_1 \smile \beta + \alpha_2 \smile \beta, [v_0, \ldots, v_{p+q}] >$

and

$< \alpha \smile (\beta_1 + \beta_2), [v_0, \ldots, v_{p+q}] > =< \alpha, [v_0, \ldots, v_p] > . < (\beta_1 + \beta_2), [v_p, \ldots, v_{p+q}] >$

$=< \alpha, [v_0, \ldots, v_p] > .(< \beta_1, [v_p, \ldots, v_{p+q}] > + < \beta_2, [v_p, \ldots, v_{p+q}] >)$

$=< \alpha, [v_0, \ldots, v_p] > . < \beta_1, [v_p, \ldots, v_{p+q}] > +$

$+ < \alpha, [v_0, \ldots, v_p] > . < \beta_2, [v_p, \ldots, v_{p+q}] >$

$=< \alpha \smile \beta_1, [v_0, \ldots, v_{p+q}] > + < \alpha \smile \beta_2, [v_0, \ldots, v_{p+q}] >$

$=< \alpha \smile \beta_1 + \alpha \smile \beta_2, [v_0, \ldots, v_{p+q}] >,$

we get $(\alpha_1 + \alpha_2) \smile \beta = \alpha_1 \smile \beta + \alpha_2 \smile \beta$ and $\alpha \smile (\beta_1 + \beta_2) = \alpha \smile \beta_1 + \alpha \smile \beta_2$. □

Theorem 6.3. $\delta(c^p \smile c^q) = \delta c^p \smile c^q + (-1)^p c^p \smile \delta c^q$.

Proof. We say that the values of the digital simplicial cochains $\delta c^p \smile c^q$ and $(-1)^p c^p \smile \delta c^q$ at $[v_0, \ldots, v_{p+q+1}]$ are equal to

$$\sum_{0 \leq i \leq p} (-1)^i c^p[v_0, \ldots, \hat{v_i}, \ldots, v_{p+1}] c^q[v_{p+1}, \ldots, v_{p+q+1}] \qquad (1)$$

and

$$(-1)^p \sum_{p \leq i \leq p+q+1} (-1)^{i-p} c^p[v_0, \ldots, v_p] c^q[v_p, \ldots, \hat{v_i}, \ldots, v_{p+q+1}], \qquad (2)$$

respectively. The first term in (2) removes the last term in (1). The sum of the other terms in these sums equals the value of the digital simplicial cochain $\delta(c^p \smile c^q)$ at $[v_0, \ldots, v_{p+q+1}]$. □

Theorem 6.4. *Let (X, κ) be a digital simplicial complex. The cup product on digital simplicial cochains is associative, that is,*

$$(c^p \smile c^q) \smile c^r = c^p \smile (c^q \smile c^r).$$

The digital simplicial cochain given by 1_X is the unit element, that is,

$$1_X \smile c^p = c^p \smile 1_X = c^p.$$

Proof. Let $c^p \in H^{p,\kappa}(X, G_1)$, $c^q \in H^{q,\kappa}(X, G_2)$ and $c^r \in H^{r,\kappa}(X, G_3)$. Then

$$< (c^p \smile c^q) \smile c^r, [v_0, \ldots, v_{p+q+r}] > =< (c^p \smile c^q), [v_0, \ldots, v_{p+q}] > \\ . < c^r, [v_{p+q}, \ldots, v_{p+q+r}] >$$

$$= (< c^p, [v_0, \ldots, v_p] > . < c^q, [v_p, \ldots, v_{p+q}] >). < c^r, [v_{p+q}, \ldots, v_{p+q+r}] >$$

$$=< c^p, [v_0, \ldots, v_p] > .(< c^q, [v_p, \ldots, v_{p+q}] > . < c^r, [v_{p+q}, \ldots, v_{p+q+r}] >)$$

$$=< c^p, [v_0, \ldots, v_p] > .(< c^q \smile c^r, [v_{q+r}, \ldots, v_{p+q+r}] >)$$

$$=< c^p \smile (< c^q \smile c^r), [v_0, \ldots, v_{p+q+r}] > .$$

For the other part, we get

$$< 1_X \smile c^p, [v_0, \ldots, v_p] > =< 1_X, [v_0, \ldots, v_p] > . < c^p, [v_0, \ldots, v_p] > \\ =< c^p, [v_0, \ldots, v_p] >$$

and similarly

$$< c^p \smile 1_X, [v_0, \ldots, v_p] > =< c^p, [v_0, \ldots, v_p] > . < 1_X, [v_0, \ldots, v_p] > \\ =< c^p, [v_0, \ldots, v_p] > .$$

□

Theorem 6.5. *If $c^p \in H^{p,\kappa}(X, G_1)$ and $c^q \in H^{q,\kappa}(X, G_2)$ are cocycles, then*

$$c^p \smile c^q = (-1)^{pq} c^q \smile c^p.$$

Proof. We obtain

$$< c^p \smile c^q, [v_0, \ldots, v_{p+q}] >=< c^p, [v_0, \ldots, v_p] > . < c^q, [v_p, \ldots, v_{p+q}] >$$

and

$$< c^q \smile c^p, [v_{p+q}, \ldots, v_0] >=< c^q, [v_{p+q}, \ldots, v_p] > . < c^p, [v_p, \ldots, v_0] > .$$

Note that

$$[v_r, \ldots, v_0] = (-1)^{r(r+1)/2}[v_0, \ldots, v_r] \quad \text{and} \quad (p+q)(p+q+1)-p(p+1)-q(q+1) = 2pq.$$

□

Theorem 6.6. *Let $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images. If $f : (X, \kappa_1) \longrightarrow (Y, \kappa_2)$ is a digitally continuous map and $c^p \in H^{p,\kappa}(X, G_1)$ and $c^q \in H^{q,\kappa}(X, G_2)$ are cocycles, then*

$$f^*(c^p \smile c^q) = f^*(c^p) \smile f^*(c^q).$$

Proof. We get

$$\begin{aligned} < f^*(c^p \smile c^q), [v_0, ..., v_{p+q}] > &=< c^p \smile c^q, [f(v_0), ..., f(v_{p+q})] > \\ &=< c^p, [f(v_0), ..., f(v_p)] > . < c^q, [f(v_p), ..., f(v_{p+q})] > \\ &=< f^*(c^p), [v_0, ..., v_p] > . < f^*(c^q), [v_p, ..., v_{p+q}] > \\ &=< f^*(c^p) \smile f^*(c^q), [v_0, ..., v_{p+q}] > . \end{aligned}$$

□

Corollary 6.7. The digital cup product gives a ring structure to cohomology groups of digital simplicial complexes.

7. Conclusion

The goal of this paper is to determine properties which don't hold for digital topology. We state the Eilenberg-Steenrod axioms for the simplicial cohomology groups of digital images. Particularly we present that excision and homotopy axioms don't hold for digital images. We show that the Künneth formula for digital simplicial cohomology need not to hold for digital images. The digital simplicial cohomology is required to research the digital cohomology operations which is very important in homotopy theory.

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