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On Zadeh's Contribution to the Optimality Problem: Four Points of View on Optimality – Logic, Set Theory, Theory of Uncertainty, and Analytic

In memoriam Lotfi A. Zadeh

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Abstract. We revisit the already old problem of optimality, focusing on Zadeh's contributions, in the context of the developments in the last 50 years. In doing so, we recall and analyze a major direction of contribution of Zadeh, which was little noticed in studies on his work, moreover is not clearly stressed in his own articles. Specifically, we posit that Zadeh largely reinvented the notion of optimality and offered new interpretations to this concept. For clarifying the context, we suggest a generalized view of the optimality problem. Opened problems are listed.

Key-words: optimality, imprecision, limited information, fuzzy logic, interpolator, universal approximator.

On September 6, 2017, a great scientist and engineer passed away – Lotfi A. Zadeh, the brain-father of fuzzy logic, fuzzy systems, and soft computing theories. Zadeh is mostly known for having invented fuzzy set theory and fuzzy logic. One tends to forget two of his major contributions, the Z transform, he and Ragazzini jointly proposed, and his contributions to the optimality theory. Everyone having a cellular phone or a computer is indirectly using the Z transform, which is fundamental in all procedures for audio and image processing. As a tribute to Zadeh's memory, this article revisits the optimality problem and places Zadeh's contribution in a general frame.

We suggest that there are at least four distinct approaches to optimality in decision making and subsequently in game theory, control, classification, and signal processing.

The set theory approach is focused on representing sets of items with specific features and on selecting a set or a collection of sets that, from some point of view, is preferable to the others.

Clustering and classification are directly connected to this first approach.

The logical approach is seemingly more connected with the decision theory and with algebraic treatments; among others, it emphasizes the possibility that no preference can be selected that is consistent (no order of preferences). 'If-Then' rules are often used in the logic-type approach. Fuzzy logic operators, among others, are well formalized using the concept of triangular norms (t-norm and s-conorm; for details see for example [1]; yet, Zadeh was not much attracted by these concepts and I am not aware of the use of these concepts in any substantial way in Zadeh's works. In the context of logic and decision, optimality is arbitrary because there is no way to decide on the choice of the type of logic to deal with the problem in hand; that is, there is no way to choose the t-norms and conorms for defining the logic operators. The various types of fuzzy logics (min-max, max-product) engineers applied in numerous studies illustrate the issue. In fact, many studies in optimal fuzzy control analyzed the optimal choice of the type of fuzzy logic that, for specified problems, leads to optimal solutions.

As logics and the theory of sets are algebraically equivalent (Heyting algebra for fuzzy logic, see [2] for fundamentals, Boolean algebra for the standard set theory and binary logic etc.), the corresponding two approaches can be seen as equivalent; consequently, jumping from one to the other is permissible during discussion. Zadeh was aware of this equivalence, but has not used it; he wrote distinct papers on fuzzy sets [3] and fuzzy logic [4, 5], not connecting them through the concept of algebras.

The approach based on uncertainty theories, stochastic and fuzzy-type based, emphasizes on imprecision and limited knowledge. On this realm, statisticians and fuzzy set theorists (including Zadeh) clashed. The uncertainty in standard stochastic approaches can be related to sets and features that have assigned probability distributions. Zadeh's proposal was to assign to them degrees of membership [3-6]; next, he proposed a theory of possibilities that was further developed by Prade and Dubois [7].

The fourth approach is analytic; it is connected to descriptions in metric spaces, to general mathematical analysis, and to other branches, such as graph and network theories; optimal control, optimal flow, optimal choices for the parameters of systems are all related to this approach, which is the most popular, especially in engineering and operational research applications. Zadeh, in a brief article suggestively titled "*What is optimal*" pinpointed several difficulties in this approach. One of these difficulties could be stated (slightly enlarged) as: Because there is an infinite number of distances on $\mathbf{R}^n$, all equally justified, there is an infinite number of optimality criteria and no reason to prefer one of them to the others. In common words, something is optimal or not, depending of what is the goal.

Subsequently, we provide more factual details on Zadehs studies on optimality. In a series of papers [8-14], Zadeh made a critical analysis of the optimality concept, including Wiener optimal filtering, and studied various optimality problems. Years later, he also suggested the use of fuzzy logic as a tool to deal with both optimality and uncertainty [8]. He has not, however, provided more than an intuitive discussion of the subject of using fuzzy logic and fuzzy systems in optimality.

The optimality preoccupied Zadeh since the end of the 1940s and the beginning of 1950s. His very first paper he deemed notable and thus listed in the "Publications List. Lotfi A. Zadeh" [15], deals with the topic. But Zadeh did not consider it highly important because in the head of the list he wrote "Principal papers are highlighted", yet that paper is not. Precisely, the paper "Probability Criterion for the Design of Servomechanisms" by John R. Ragazzini and Lotfi A. Zadeh, Journal of Applied Physics, 1949 [8] addresses finding an optimality design criterion based not on least square errors but of the "maximization of the probability" that the absolute error is larger than a specified threshold, with constraints related to "*physical realizability of the system*." After performing the analysis, they found that in many practically important cases "*resultant design is the same as that which would be obtained by the use of the minimum meansquare error criterion*". This conclusion is reassuring, because it provides a sense of equivalence between at least two different optimality criteria and optimality solutions. The topic of this type of equivalence was not pursued up till now and could be a significant track of study in the future.

Another paper he published on the topic, together with J. R. Ragazzini, in 1952 [10], was devoted to optimal filtering ("Optimum Filters for the Detection of Signals in Noise"). The paper addresses

"An optimum predetection filter is defined in this paper as one which maximizes the "distance" between the signal and noise components of the output (subject to a constraint on the noise component) in terms of a suitable distance function d(x, y)." Their contribution is: *"North's theory of such filters is extended to the case of nonwhite noise and finite memory (i.e., finite observation time) filters"*[10].

Next, in his brief article on optimization, published in IRE Transactions on Information Theory, p. 3, 1958, *What Is Optimal* [12], Zadeh essentially argues that there is no applicable criterion for choosing the optimality criterion in the definition of optimality; therefore, he concludes, the notion of optimality is illusive. He argues that scalar criteria of optimality are subjective or ad-hoc adopted, while vectorial criteria are often intransitive or otherwise inconsistent. He seems to suggest that the notion of 'optimality' and the search for 'optimality criteria' should be abandoned and replaced with the notion of constraints, but fails to see that the constraints are either arbitrarily chosen or chosen to satisfy some 'benefit', the latter suffering of the same issues as 'optimality'. Nevertheless, Zadeh's brief article on optimality is still actual and should be re-read and its critic applied to all newly proposed optimality criteria and related methods. The topic is recurrent in Zadeh's works and discussed in technical terms, in 1963, in the paper "Optimality and non-scalar-valued performance criteria" [14].

Interestingly, in Zadeh's fundamental paper "Fuzzy Sets and Systems" [16], one of the very first examples he gives for fuzzy classes (sets) is the set of adaptive systems, which directly connects to optimization and optimal systems, *"The notion of fuzziness . . . relates to situations in which the source of imprecision is . . . classes which do not possess sharply defined boundaries, e.g., the 'class of adaptive systems'. . . "* Nevertheless, he may have failed to see that fuzzy logic systems (FLSs) are so successful in control modeling, and optimization applications because they use a 'greedy' approach of local and piecewise interpolation and approximation. Zadeh seems to pay no attention to the fact that fuzzy logic systems with defuzzifiers are mappings from $\mathbf{R}^n$ to $\mathbf{R}$ ($\mathbf{R}^n \to \mathbf{R}$) and that the attainment of FLSs in many applicative fields comes from this basic fact. Zadeh never mentioned key properties of the FLSs such as the properties of exact interpolators and universal approximators [17] – fundamental properties that actually guarantee the very usefulness of FLSs.

The concept of optimality can be defined broadly as follows. Given

a) A logic *L* for making decisions on the optimality (solution quality);

b) A set of criteria *C* for assessing the quality of the decision and a meta-criterion determining how the criteria concatenate; the meta-criterion may be "maximizes the weighted sums of scores according to the criteria", or "satisfies the higher number of individual criteria" or "selects the highest level of the combinations of vector of criteria, in a ordered set of combination" , or some other type of multi-objective optimization;

c) A set of items to optimize, *J*, which can be models, systems, networks, classifier, or decisions under a set of circumstances;

d) A set of circumstances, *E*, for example, environment conditions, noise etc. and inputs to the system;

e) A set of constraints, *K*; then

Select from *J* the subset of items (possibly a single one, or none) that obey the constraints *K* and satisfies the meta-criterion for conditions *E*, under the given logic.

Notice that when any of the elements (b) to (e) are expressed in fuzzy logic terms, still the logic of decision on the optimality, (a), may be different from the ones used in (b)– (e); multiple logics can be combined in the optimality problem. In their early papers, Zadeh and Ragazzini and independently Zadeh shed light on aspects related to the parts (b)-(e) of the above general description of optimality. It is somewhat surprising that Zadeh has not addressed, with the tools offered by fuzzy logics, the issue denoted by (a) above. It would be stimulating to compare, for example, solutions under possibilistic and probabilistic settings, following ideas such as in [17]. An ingenious proposal by Zadeh in his paper Fuzzy Sets and Systems was to deal with fuzzy constraints, especially when they are motivated by the fact that in many practical optimization problems, particularly these involving man-machine systems, the constraints on variables are seldom sharply defined. Unfortunately, this reach in consequences idea was not charted enough after that paper, neither by Zadeh nor by other authors. This may be due to the fact that Zadeh suggested that the problem is easily reducible to one of unconstrained optimization. Specifically, he shows that, for the problem of maximization of the objective function f(**x**) over a fuzzy constraint **x** belongs to the fuzzy set A, the problem reduces to an unconstrained maximization of the function $f^*(\mathbf{x})=f(\mathbf{x}).\ \mu_A(\mathbf{x})$, with the criterion directly changed by multiplication with the membership function of the fuzzy constraint. Further, analyzing the case when $\mathbf{x}=(x_1, \ldots, x_k, \ldots, x_n)$ and x_k have the constraints 'x_k belongs to the fuzzy set A_k', but he provides only an approximate solution and is ends on an unenthusiastic note, "*There are many cases in which optimization under fuzzy constraints is ineffective or computationally infeasible as an alternative to conventional optimizatio* ..." Yet, numerous other studies proved that methods pertaining to fuzzy optimization are powerful and applicable to a large range of problems, for example [18-20].

Zadeh also introduced the notion of *Z*–number [21] as a conceptual tool in decision making. The use of Z numbers in optimization was recently analyzed in [22, 23], among others, but remains largely unused to its full potential. Also, it might be very interesting to find under what conditions solutions to optimality under one set of circumstances and criteria become optimal under another set circumstances and criteria when the items in the first set of items to optimize are transformed by some mapping, for example by a continuous and monotonic transform. The method was successfully used in many other problems.

A more general optimization problem is: Select from a set C^* of sets of criteria the ones that, for the given *L, J, E, K,* produce the same subset of *J*. When two sets of criteria produce no common selection of items, they may be named incompatible. When two sets of criteria produce the same selection of items, they are equivalent. The second problem was in a small degree tackled by Zadeh and Ragazzini, but both problems remain largely untouched and unsolved today. Surprisingly, even Zadeh has not recalled his contributions to the optimality problem in his overview of his work on fuzzy logic and systems [24].

It is worth mentioning that in his scientific self-biography "Fuzzy logic – a personal perspective", he mentioned the words related to "optimal" only once. He says "*More concretely, during the period 1950–1960, my research was concerned, in the main, with systems analysis, optimization and information systems.*"

Concluding, Zadeh's contribution to the understanding, definition, treatment, and application of optimality is an illuminating one and the multiple avenues opened by him in optimality theory should be continued, possibly in the framework of the AI and 'intelligent systems' domains. Zadeh's views on optimality should be revisited in detail in the future and the progresses he made on the topic of optimality added to the long list of his contributions to the science of the second half of the 20th century. Zadeh's work spurred a huge amount of research ranging from mathematics to social sciences, moreover implanted numerous major opened problems that promise significant new results in several domains, including that of optimality and its numerous branches.

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