**ROMANIAN JOURNAL OF INFORMATION SCIENCE AND TECHNOLOGY**
Volume **24**, Number 3, 2021, 257–283

# Fuzzy Modus Ponens and Tollens based on Moving Distance in SISO Fuzzy System

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**Abstract.** We show that this paper points out a basic and original fuzzy reasoning method that can draw a novel study direction of the approximate inference in fuzzy systems with uncertainty. For realization of this study direction this work is based on the recent paper's idea presented by the several authors, which is to obtain a new conclusion by the moving distance operation between the antecedent and the given premise (observation). Firstly in this paper we propose a criterion function for checking of the reductive property about fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT). Secondly unlike fuzzy reasoning methods based on a lot of the similarity measure, we propose a principle of new FMP and FMT based on moving distance (MD), i.e., moving operation, and then present two theorems and reasoning methods for FMP-MD and FMT-MD, for short, MDM. Thirdly through the several computational experiments, we show that proposed method is simple and effective, has high reductive property, and is better in accordance with human thinking than existing methods.

**Key-words:** Fuzzy Reasoning: Moving Distance: Fuzzy Modus Ponens: Fuzzy Modus Tollens: Normal Fuzzy Set: Reductive Property: SISO Fuzzy System

## 1. Introduction and preliminary results

This paper points out a basic and original fuzzy reasoning method, i.e., new fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) that can draw a novel study direction in the approximate inference with uncertainty. FMP and FMT are two fundamental patterns of general fuzzy reasoning [43]. And reductive property is one of the essential and important properties in the applications of the fuzzy inference mechanism [21, 41]. Since the inception of Zadeh's pioneering paper [43], the reductive property of the fuzzy reasoning method in the fuzzy inference systems (FIS) has regarded as an very important topic in the fuzzy theory and its applications [9, 21, 31, 39]. In [43] Zadeh proposed the Compositional Rule of Inference (CRI) for FMP and

FMT. In [37] Wang presented the Triple Implication Principle (TIP) with total inference rules of fuzzy reasoning. Since the inception of the triple I method [37], many papers have researched the fuzzy inference method. In [45] the quintuple implication principle (QIP) for solving FMP and FMT was proposed unlike Zadeh's CRI [42] and Wang's TIP [21]. In [34] concept of the fuzzy similarity measure is axiomatically defined, which is a generalization of the classical similarity measure (SM) and then fuzzy similarity inference (FSI) is proposed. Several formulas are presented to calculate SM of two fuzzy sets. And in [34], computational formulas for $\alpha$-FSI FMP and $\alpha$-FSI FMT are proposed on the basis In [2, 23, 36], and then the reductive property, i.e., the reversibility of the proposed FSI is discussed. In [38] author pointed out that many papers have been done for computing and analyzing the fuzzy inference conclusion $B^*$ which are valuable for solving problems in fuzzy control and are meaningful with respect to theoretic aspect. In [42] reverse triple I method of fuzzy reasoning was proposed. In [13] based on Schweizer-Sklar operators new triple I algorithms was presented. In [42] based on Lukasiewicz implication operator the reverse triple I method was proposed. In [44] the triple I method of fuzzy reasoning based on intuitionistic fuzzy set is presented. However in [10] shortcoming of the triple I method is pointed out, which cannot be applied in fuzzy control. There are a lot of the fuzzy inference method based on similarity measures. In [31, 32], the shortcomings of CRI method are mentioned, so a similarity-based fuzzy reasoning method called approximate analogical reasoning schema (AARS) is proposed. In [4, 5] in order to solve the problems in medical diagnosis, two similarity-based fuzzy reasoning methods are proposed. In [40] three fuzzy reasoning methods based on similarity (i.e., IC based on inclusion and cardinality, DS based on degree of subsethood, and EC based on equality and cardinality) are proposed. The principle of fuzzy reasoning methods based on similarity is to obtain new fuzzy set $B^*$ of the inference result by modifying fuzzy set $B$ of the consequent with a modification function based on the similarity between $A$ and $A^*$ . Compared with the CRI method, the fuzzy reasoning methods based on similarity do not require the calculation of fuzzy relation. However the results obtained by these methods, strongly depend on the similarity measure and the modification function. In [20] author pointed out that the fuzzy reasoning methods based on the fuzzy relation $R_m, R_a, R_c$ and $R_p$ do not satisfy the reductive property, but they can be applied to the practical problem, for example fuzzy control. In [17-20] the authors mentioned that the reasoning methods based on the fuzzy relation $R_{ss}, R_{sg}, R_s, R_{gg}, R_{gs}$ and $R_g$ do satisfy the reductive property, but they cannot be applied to the practical problem, for example fuzzy control. That is, as mentioned in [20], this is a contradict. The experimental fuzzy control results are presented in [20]. To overcome this contradiction between the reductive property and fuzzy control, fuzzy reasoning method based on a new principle must be developed, so far, a lot of fuzzy reasoning methods were proposed, studied, checked, and applied in many branches. There are a lot of applications of fuzzy theory. For example, in 2011 [1], authors presented a novel hybrid fuzzy model based expert system for misfire detection in automobile engines. And in 2017, [22, 33] shows a new medical image retrieval method based on fuzzy S-tree and vector quantization. And in 2018 [8], authors pointed out surrogate model based optimization method of traffic lights cycles and green period ratios based on fuzzy rule interpolation and microscopic simulation, and the generation of a fuzzy model is presented and a methodology is suggested, where the fuzzy system is applied as a surrogate model for the determination of the optimal green period ratios and traffic light cycle times. In 2019, [28] presented a novel combined model-free adaptive control method with fuzzy component based on virtual reference feedback tuning for tower crane systems, which is based on [24–27, 29]. In [28] a novel mix of 2 data-driven algorithms is proposed. The mix algorithms aims to exploit the main advantage of data-driven virtual reference feedback tuning (VRFT) algorithm, which is represented by the automatic computation of the optimal parameters using a metaheuristic grey wolf optimizer (GWO) for the compact form dynamic linearization (CFDL) version of [29]'s model-free adaptive control based Takagi-Sugeno's fuzzy algorithm (CFDLPDTSFA), thus the parameters of the CFDL-PDTSFA are optimally tuned in a model-free type via VRFT. The 3 specific optimization problems are defined and accomplished by model-free adaptive control, VRFT and GWO based algorithms. The new result is validated by using experimental results to the arm angular position of the nonlinear tower crane system laboratory equipment. Unlike CRI [43], TIP [10, 12, 37, 41, 42, 44], and FSI [31, 32, 34, 39], In [14] authors presented a new fuzzy modeling method for industry furnace's real-time temperature prediction, which is based on moving of fuzzy membership. And In [15], authors on the basis [14], proposed a new fuzzy modus ponens and fuzzy modus tollens based on the compensating fuzzy reasoning (CFR). This method is called a modus ponens and fuzzy modus tollens based on CFR. Its principle is to acquire a new fuzzy reasoning conclusion by moving and deforming of the consequent fuzzy set in response to compensating operation, i.e., the moving, deformation, and moving-deformation operations between the antecedent fuzzy set and the given premise. It is not only in accordance with human thinking but also satisfies the reductive property of the fuzzy reasoning. The proposed method [42] is not based on the traditional implication and its operation is simpler than the previous reasoning methods [2, 4, 7, 15, 16, 21, 32]. This method can be applied in a lot of application branches such as system identification, fuzzy control, image processing, decision making, fuzzy prediction and so on. In this paper on the basis of the paper [15] we present a new criterion function for checking of the reductive property about the fuzzy reasoning result for fuzzy modus ponens and fuzzy modus tollens. And then, unlike fuzzy reasoning methods based on the similarity measure, we propose a new fuzzy reasoning method based on moving distance (MD) presented in [31,32], i.e., moving operation in the paper [15]. And we compare the reductive properties for 5 fuzzy reasoning methods, concretely 17 individual methods, with respect to FMP and FMT, which are CRI, TIP, QIP, AARS, and our new MD method. We show that MD based method, for short, MDM proposed in this paper is better in accordance with human thinking than CRI, TIP, QIP, and AARS. The rest of this paper is organized as follows. In section 2, we discuss backgrounds for FMP and FMT about the fuzzy reasoning methods based on fuzzy implication relation. In section 3, a new criterion function and fuzzy reasoning method are presented, respectively. In section 4, the reductive property of CRI, TIP, QIP, AARS and our proposed MDM, concretely, 17 individual fuzzy reasoning methods are all checked by MATLAB computational experiments. In section 5 conclusion of this paper is shown.

## 2. Backgrounds

Generally known fuzzy reasoning methods are FMP and FMT in the fuzzy system with 1 input 1 output 1 rule. General form of FMP presented in [5,45] is as follows.

Rule; *if* $x$ *is* $A$ *then* $y$ *is* $B$, Premise: $x$ *is* $A^*$, Conclusion: $y$ *is* $B^*$ (1)

General form of FMT in the paper [6,45] is as follows.

Rule; *if* $x$ *is* $A$ *then* $y$ *is* $B$, Premise: $y$ *is* $B^*$, Conclusion: $x$ *is* $A^*$ (2)

, where $A^* \in F(X), A \in F(X)$ are fuzzy sets defined in the universe of discourse $X$, $B^* \in F(Y), B \in F(Y)$ are fuzzy sets defined in the universe of discourse . In the fuzzy system with 1 input 1output n rules, we rewrite the definition for reductive property of fuzzy inference method in [6]. According to [12], the formula (2) can be written as follows, because FMT is opposite to FMP.

$$\text{Rule; } \textit{if } y \textit{ is } \overline{B} \textit{ then } x \textit{ is } \overline{A}, \text{ Premise: } y \textit{ is } \overline{B}, \text{ Conclusion: } x \textit{ is } \overline{A} \tag{3}$$

where $\overline{A} = 1 - A, \overline{B} = 1 - B$. For the formula (1), (2), and (3), according to Zadeh's viewpoint, Rule is represented by some fuzzy relation. For example, operator $\rightarrow_z$ is Zadeh's implication, the fuzzy relation of the Rule is presented as follows.

$$R(x, y) = A(x) \rightarrow_z B(y), \quad a \rightarrow_z b = (1 - a) \vee (a \wedge b) \tag{4}$$

In [45], authors listed 4 most important implication operators and the corresponding t-norms. As mentioned in [45], Łukasiewicz's implication $a \rightarrow_L b$ and the corresponding t-norm $a \otimes_L b$, gödel's $a \rightarrow_G b$ and $a \otimes_G b$, $R_0$'s $a \rightarrow_{R_0} b$ and $a \otimes_{R_0} b$, and Gougen's $a \rightarrow_{G_0} b$ and $a \otimes_{G_0} b$ are described as follows, respectively.

$$a \rightarrow_L b = 1 \wedge (1 - a + b), \quad a \otimes_L b = 0 \vee (a + b - 1) \tag{5}$$

$$a \rightarrow_G b = \begin{cases} 1, if a \leq b \\ b, if a > b \end{cases}, a \otimes_G b = a \wedge b \tag{6}$$

$$a \rightarrow_{R_0} b = \begin{cases} 1, if a \leq b \\ a' \vee b, if a > b \end{cases}, a \otimes_{R_0} b = \begin{cases} 0, if a + b \leq 1 \\ a \wedge b, if a + b > 1 \end{cases}, a' = 1 - a \tag{7}$$

$$a \rightarrow_{G_0} b = \begin{cases} 1, if a \leq b \\ \frac{b}{a}, if a > b \end{cases}, a \otimes_{G_0} b = ab \tag{8}$$

# 3. New Reductive Property Criterion Function and Fuzzy Reasoning Method

## 3.1. Motivation and Importance of New Fuzzy Reasoning Method

### 3.1.1. Motivation

The reductive property is one of the essential and important properties in the fuzzy reasoning [21, 41]. But a lot of fuzzy reasoning methods have some shortcomings. The motivations obtained from some shortcomings are as follows:

- As mentioned in [31, 32], the underlying semantic of CRI is unclear, its reasoning result does completely not satisfy the reductive property. Therefore reasoning method that does satisfy the reductive property must be studied.

- As pointed out in [10], the shortcoming of TIP is that it cannot be applied in fuzzy control. Therefore fuzzy reasoning method that can be applied in fuzzy control must be researched.
- As presented in [20], the some fuzzy reasoning methods based on the fuzzy relation have the contradict that they can be applied to the practical problem, for example fuzzy control, but do not satisfy the reductive property, vice versa. Therefore fuzzy reasoning method that has not some contradict must be studied.
- As mentioned in [33], the fuzzy reasoning methods based on similarity measure (SM) depend strongly on the similarity measure and the modification function, and do not completely satisfy the reductive property. Therefore fuzzy reasoning method that does not depend on the similarity measure and the modification function must be researched.
- As presented in [37], due to many fuzzy reasoning methods based on SM do use non-linear operators, the fuzzy sets of reasoning result are non-normal and non-convex ones. Therefore in fuzzy reasoning processing, linear operators must possibly be used.
- According to [3, 7, 16–19], a lot of fuzzy reasoning methods mathematically seem that they are all accompanied with a common shortcoming, that is, information loss. One of the reasons that do not satisfy the reductive property is to refer to losses of information occurred in reasoning processes. Therefore, information loss must possibly be reduced in fuzzy reasoning processing.
- As shown in [6, 17, 18, 21], the criterion function for checking of fuzzy reasoning results has only 2 values, i.e., '1' or 'OK' for satisfaction of the reductive property, '0' or '$\times$' for non-satisfaction of one. That is, this evaluation is too strict for the reductive property. Therefore criterion function for checking of fuzzy reasoning result must possibly be defined flexibly.

Comprehensively, in order to overcome existing shortcomings presented in [3, 6, 7, 10, 16-20, 21, 31, 32, 34, 41], fuzzy reasoning method based on new methodology or idea must be developed without some losses of information and with smooth evaluation for the reductive property. From the above mentioned facts, we try to develop a new fuzzy reasoning method with respect to the information loss and reductive property. This is a motivation of this paper.

#### 3.1.2. Possibility and Importance

In a lot of papers fuzzy reasoning methods based on SM are proposed. Their basic idea is to consider the similarity measure of the consequent $B(y)$ and the fuzzy reasoning conclusion $B^*(y)$ if the antecedent $A(x)$ is similar to the given premise $A^*(x)$ for FMP. This idea is right. By the way we consider following:

- Similarity measure and moving distance have inverse proportional relation. That is, "the antecedent $A(x)$ is similar to the given premise $A^*(x)$" is approximately equal to "the antecedent $A(x)$ is closer to the given premise $A^*(x)$". Here "similar" is correspondent to similarity measure, "closer" to moving distance. if $A(x)$ is completely equal to $A^*(x)$ then the similarity measure is 1 and moving distance is 0.
- The fuzzy reasoning methods based on similarity do not require the calculation of fuzzy relation or implication. However the fuzzy reasoning results obtained by the similarity methods depend strongly on the similarity measure and the modification function.

- The fuzzy reasoning methods based on similarity do use nonlinear, i.e., max, min operator. Thus fuzzy reasoning methods based on similarity measure have a lot of information loss [37]. But fuzzy reasoning methods based on moving distance [14,15] can be used linear operator for example summation and subtraction, thereby information loss can be reduced.
- And the similarity measure has closed interval $[0, 1]$ and moving distance $[0, m]$, where $m$ is a finite number, $m > 0$. According to these facts, fuzzy reasoning based on moving distance (MD) is possible. This is a possibility and an importance of our paper.

## 3.2. Reductive Property Criterion Function

The reductive property is one of the essential properties in the applications of the fuzzy inference mechanism [6,19] . According to [19,39], four cases of the premise for FMP in Class 1 are as follows; Case 1: $A^*$ *is* $A$, Case 2: $A^*$ *is very* $A(= A^2)$, Case 3: $A^*$ *is more or less* $A(= A^{1/2})$, Case 4: $A^*$ *is not* $A(= 1-A)$. Since FMT is opposite to FMP, according to [19,39], four cases of the given premise for FMT in Class 1 are as follows; Case 6: $B^*$ *is not* $B(= 1-B)$, Case 7: $B^*$ *is not very* $B(= 1 - B^{1/2})$, Case 8: $B^*$ *is not more or less* $B(= 1 - B^{1/2})$, Case 9: $B^*$ *is* $B$. And four cases of the Premise for FMP in Class 2 are as follows; Case 1: $A^*$ *is* $A$, Case 2: $A^*$ *is very* $A(= A^2)$, Case 3: $A^*$ *is more or less* $A(= A^{1/2})$, Case 5: $A^* =$ *slightly titled of* $A(= s.t.\ A)$. And four cases of the given premise for FMT in Class 2 are as follows; Case 6: $B^*$ *is not* $B(= 1 - B)$, Case 7: $B^*$ *is not very* $B(= 1 - B^2)$, Case 8: $B^*$ *is not more or less* $B(= 1 - B^{1/2})$, Case 10: $B^* =$ *slightly titled of* $B(= s.t.\ B)$.

What conclusion $B^*$ for FMP and $A^*$ for FMT can be obtained? For this, Table 1 shows reductive property of FMP and FMT based on [6] and [21]. In Table 1, Case 1, Case 2, Case 3, and Case 4 for FMP, Case 6, Case 7, Case 8, and Case 9 for FMT are criterion functions based on the paper [21], and Case 1, Case 2, Case 3, and Case 5 for FMP, Case 6, Case 7, Case 8, and Case 10 for FMT are criterion functions based on the paper [6]. In the paper [6], authors mentioned that their proposed method is based on the assumption that the premise $A^*$ is slightly different from the antecedent of fuzzy rule $A$ and thus the conclusion $B^*$ is slightly different from the consequent $B$ of fuzzy rule, therefore, they do not expect a reasonable conclusion if the premise $A^*$ is different from the antecedent $A$ too much. Unlike the classical reasoning, if the given premise $A^*$ is not exactly equal to the antecedent $A$, then we can still obtain fuzzy reasoning result $B^*$.

However we know that if the given premise $A^*$ and the antecedent $A$ are totally different, then the fuzzy reasoning result $B^*$ might be unreasonable or uninformative. Then in practical applications, a group of fuzzy rules called rule base is used to avoid the incorrect fuzzy reasoning result caused by the deviation between the given premise $A^*$ and the antecedent $A$. As obviously mentioned in the paper [11], if the given premise $A^*$ is slightly different from the antecedent $A$ then the fuzzy reasoning conclusion $B^*$ is slightly different from the consequent $B$. According to combination of the paper [6] and [20], for example the antecedent fuzzy set $A = [small]$ and consequent fuzzy set $B = [large]$, we can obtain the following Table 1. In Table 1 Class 1 and Class 2 are as follows.

**Table 1.** New reductive property criterion for FMP and FMT based on [6,21]

| FMP | *if $x$ is $A$ then $y$ is $B$* | | |
|---|---|---|---|
| | $x$ *is* $A^*$ [6,21] | $y$ *is* $B^*$ [6,21] | New reductive property criterion $RPCF_{FMP}$ of $y$ *is* $B^*$, (%) |
| Case 1 | $A^*$=$A$ | $B^*$=$B$ | (1-$\sum_{k=1}^{r}$\|$b_{kl}^*$-$b_k$\|/$r$)× 100 |
| Case 2 | $A^*$=$A^2$ | $B^*$=$B^2$ or $B$ | (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-$b_k^2$\|/$r$)× 100 or (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-$b_k$\|/$r$)× 100 |
| Case 3 | $A^*$=$A^{\frac{1}{2}}$ | $B^*$=$B^{\frac{1}{2}}$ or $B$ | (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-$b_k^{\frac{1}{2}}$\|/$r$)× 100 or (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-$b_k$\|/$r$)× 100 |
| Case 4 | $A^*$=1-$A$ | $B^*$=1-$B$ | (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-(1-$b_k$)\|/$r$)× 100 |
| Case 5 | $A^*$=$s.t.\ A$ | $B^*$=$s.t.\ B$ | (1-$\sum_{k=1}^{r}$ \|$b_{kl}^*$-$s.t.\ b_k$\|/$r$)× 100 |
| FMT | *if $y$ is $\overline{B}$ then $x$ is $\overline{A}$* | | |
| | $y$ *is* $B^*$ [6,21] | $x$ *is* $A^*$ [6,21] | New reductive property criterion $RPCF_{FMT}$ of $x$ *is* $A^*$, (%) |
| Case 6 | $B^*$=1-$B$ | $A^*$=1-$A$ | (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-(1-$a_k$)\|/$r$)× 100 |
| Case 7 | $B^*$=1-$B^2$ | $A^*$=1-$A^2$ or $1-A$ | (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-(1-$a_k$)$^2$\|/$r$)× 100 or (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-(1-$a_k$)\|/$r$)× 100 |
| Case 8 | $B^*$=1-$B^{\frac{1}{2}}$ | $A^*$=1-$A^{\frac{1}{2}}$ or $1-A$ | (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-(1-$a_k$)$^{\frac{1}{2}}$\|/$r$)× 100 or (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-(1-$a_k$)\|/$r$)× 100 |
| Case 9 | $B^*$=$B$ | $A^*$=$A$ | (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-$a_k$\|/$r$)× 100 |
| Case 10 | $B^*$=$s.t.\ B$ | $A^*$=$s.t.\ A$ | (1-$\sum_{k=1}^{r}$ \|$a_{kl}^*$-$s.t.\ a_k$\|/$r$)× 100 |

- Class 1; Case 1, 2, 3, and 4 for FMP, and Case 6, 7, 8, and 9 for FMT.
- Class 2; Case 1, 2, 3, and 5 for FMP, and Case 6, 7, 8, and 10 for FMT.

Since FMT is opposite to FMP, Case 1 corresponds to Case 6, Case 2 to Case 7, Case 3 to Case 8, Case 4 to Case 9, and Case 5 to Case 10, respectively. The criterion function for reductive property can be defined as the difference between the consequent of fuzzy rule and conclusion of the fuzzy reasoning. For this, several concepts based on the Table 1 are defined as follows.

**Definition 3.1** *Let fuzzy sets $A \in F(X), A_l^* \in F(X), B \in F(Y)$ and $B_l^* \in F(Y), (l = 1, 2, \cdots, s, k = 1, 2, \cdots, r)$, for FMP be their antecedent vectors $A = [a_1, a_2, \cdots, a_k, \cdots, a_r]$, the given premise vector $A_l^* = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*]$, and the consequent vector $B = [b_1, b_2, \cdots, b_k, \cdots, b_r]$. And then let the fuzzy reasoning conclusion be $B_l^* = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*]$. Then the error $E(B_l^*, B)$ between the conclusion $B_l^*$ and consequent $B$, and the error $e(A_l^*, A)$ between the given premise $A_l^*$ and the antecedent $A$ are defined as follows, respectively.*

$$\begin{aligned} E(B_l^*, B) &= [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*] - [b_1, b_2, \cdots, b_k, \cdots, b_r] \\ e(A_l^*, A) &= [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*] - [a_1, a_2, \cdots, a_k, \cdots, a_r] \end{aligned} \quad (9)$$

**Remark 3.1** In Definition 3.1, let us fuzzy sets $A \in F(X), B \in F(Y)$ and $B_l^* \in F(Y), (l = 1, 2, \cdots, s, k = 1, 2, \cdots, r)$. These fuzzy sets are called normal fuzzy sets in case that their vectors satisfy the following conditions: $A = [a_1, a_2, \cdots, a_k, \cdots, a_r] \in [0, 1]$, $A_l^* = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*] \in [0, 1]$, $B = [b_1, b_2, \cdots, b_k, \cdots, b_r] \in [0, 1]$, $B_l^* = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*] \in [0, 1]$, according to [18, 31]. For example, $A(x) = [1, 0.3, 0, 0, 0]$,

$B(y) = [0, 0, 0, 0.3, 1]$ are normal fuzzy sets. In other words normal fuzzy set include 0 and 1. In this paper we deal with normal fuzzy sets mentioned above.

**Remark 3.2** Unlike Remark 3.1, fuzzy sets are called non-normal fuzzy sets in case that their vectors satisfy the following conditions:

$A = [a_1, a_2, \cdots, a_k, \cdots, a_r] \in (0,1] \in (0,1] \quad or \in [0,1) \quad or \in (0,1), A_l^* = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*], \in (0,1] \quad or \in [0,1) \quad or \in (0,1)$

$B = [b_1, b_2, \cdots, b_k, \cdots, b_r] \in (0,1] \quad or \in [0,1) \quad or \in (0,1), B_l^* = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*] \in (0,1] \quad or \in [0,1) \quad or \in (0,1)$.

For example, $A(x) = [1, 0.3, 0.2, 0.1, 0.1]$ and $B(y) = [0.2, 0.4, 0.5, 0.7, 1]$ are non-normal fuzzy sets. In other words non-normal fuzzy set does not include 0 or 1.

**Remark 3.3** In real world, when fuzzy sets are applied, engineers and designers generally use normal fuzzy sets. In this paper we do not deal with non-normal fuzzy sets.

**Definition 3.2** *This Definition 3.2 is to generalize of the criterion for FMP shown in Table 1 according to [2,4,21-33] . The $l^{th}$ reductive property criterion function $RPCF_{FMP-FR-I}^l$ for the Case l(l=1,2,3,4, and 5, from Table 1) in FMP can be illustratively defined as the formula (10).*

$$RPCF_{FMP-FR-I}^l = \begin{cases} (1 - \sum_{k=1}^r |b_{kl}^* - b_k|/r) \times 100, & for \quad \text{Case 1} \\ (1 - \sum_{k=1}^r |b_{kl}^* - b_k|/r) \times 100, & or \\ (1 - \sum_{k=1}^r |b_{kl}^* - b_k^2|/r) \times 100, & for \quad \text{Case 2} \\ (1 - \sum_{k=1}^r |b_{kl}^* - b_k|/r) \times 100, & or \\ (1 - \sum_{k=1}^r |b_{kl}^* - b_k^{\frac{1}{2}}|/r) \times 100, & for \quad \text{Case 3} \\ (1 - \sum_{k=1}^r |b_{kl}^* - (1 - b_k)|/r) \times 100, & for \quad \text{Case 4} \\ (1 - \sum_{k=1}^r |b_{kl}^* - s.t.\, b_k|/r) \times 100, & for \quad \text{Case 5} \end{cases} \tag{10}$$

In Case 5 the given premise is $A^* = s.t.\ A$, and conclusion $B^* = s.t.\ B$. Definition 3.2 is a criterion function based on the Table 1 obtained by [6,21].

**Definition 3.3** *The reductive property criterion function $RPCF_{FMP-FR}$ for FMP of a fuzzy reasoning method (or an algorithm) is defined as follows.*

$$RPCF_{FMP-FR} = \frac{1}{S} \sum_{l=1}^{S} RPCF_{FMP-FR}^l, (\%) \tag{11}$$

**Remark 3.4** According to Definition 3.3 and Table 1, Class 1 contains Case 1, 2, 3, and 4 for FMP, and Case 6, 7, 8, and 9 for FMT, and then Class 2 contains Case 1, 2, 3, and 5 for FMP, Case 6, 7, 8, and 10, for FMT, therefore $S$ is 4 in formula (11).

**Definition 3.4** *Since FMT is opposite to FMP, let us consider the formula (3) instead of the formula (2) for FMT. Now let fuzzy sets $\overline{B} \in F(Y), B_l^* \in F(Y)$ , and $\overline{A} \in F(X)$ be antecedent vectors $\overline{B} = [1 - b_1, 1 - b_2, \cdots, 1 - b_k, \cdots, 1 - b_r]$, the given premise vector $B_l^* = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*]$, and the consequent vector $\overline{A} = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*]$, $(l = 1, 2, \cdots, s, k = 1, 2, \cdots, r)$ of fuzzy rule. And the conclusion $A_l^* \in F(X)$ be $A_l^* = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*]$, $(l = 1, 2, \cdots, s, k = 1, 2, \cdots, r)$. Then for FMT the error $E(A_l^*, \overline{A})$ between the fuzzy reasoning conclusion $A_l^*$ and consequent $\overline{A}$ of fuzzy rule, and the error $e(B_l^*, \overline{B})$ between the given premise $B_l^*$ and there antecedent $\overline{B}$ are defined as follows, respectively.*

$$E(A_l^*,\overline{A}) = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*] - [1-a_1, 1-a_2, \cdots, 1-a_k, \cdots, 1-a_r]$$
$$e(B_l^*,\overline{B}) = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*] - [1-b_1, 1-b_2, \cdots, 1-b_k, \cdots, 1-b_r] \tag{12}$$

**Definition 3.5** *The $l^{th}$ reductive property criterion function $RPCF_{FMP-FR-I}^l$ for the Case $l(l = 6, 7, 8, 9,$ and $10$ from Table 1) in FMT can be illustratively defined as follows.*

$$RPCF_{FMT-FR-I}^l = \begin{cases} (1-\sum_{k=1}^r |a_{kl}^* - (1-a_k)|/r) \times 100, & for \quad \text{Case 6} \\ (1-\sum_{k=1}^r |a_{kl}^* - (1-a_k)^2|/r) \times 100, & or \\ (1-\sum_{k=1}^r |a_{kl}^* - (1-a_k)|/r) \times 100, & for \quad \text{Case 7} \\ (1-\sum_{k=1}^r |a_{kl}^* - (1-a_k)^{\frac{1}{2}}|/r) \times 100, & or \\ (1-\sum_{k=1}^r |a_{kl}^* - (1-a_k)|/r) \times 100, & for \quad \text{Case 8} \\ (1-\sum_{k=1}^r |a_{kl}^* - a_k|/r) \times 100, & for \quad \text{Case 9} \\ (1-\sum_{k=1}^r |a_{kl}^* - s.t.\, a_k|/r) \times 100, & for \quad \text{Case 10} \end{cases} \tag{13}$$

In Case 10 the given premise is $B^*$ =*slightly titled of* $B = s.t.\ B$, Conclusion $A^* =$ *slightly titled of* $A = s.t.\ A$. Definition 3.5 is also a criterion function based on the Table 1 obtained by [6,21].

**Definition 3.6** *The reductive property criterion function $RPCF_{FMT-FR}$ for FMT are defined as follows.*

$$RPCF_{FMT-FR} = \frac{1}{S}\sum_{l=1}^{S} RPCF_{FMT-FR}^l, (\%) \tag{14}$$

The reductive property of fuzzy reasoning can be considered as the reductive property of a fuzzy reasoning method or algorithm = average of (reductive property for FMP and reductive property for FMT).

**Definition 3.7** *The criterion function for checking of the reductive property of fuzzy reasoning method is defined as arithmetic average value of $RPCF_{FMP-FR}$ and $RPCF_{FMT-FR}$ .*

$$RPCF_{FR} = \tfrac{1}{2}(RPCF_{FMP-FR} + RPCF_{FMT-FR}), (\%) \tag{15}$$

**Remark 3.5** In formula (13)-(15), indexes are the same as formula (10)-(11). According to above two definitions, when the reduction property criterion function $RPCF_{FMP-FR}$ =100(%) and $RPCF_{FMT-FR} = 100(\%)$, then the reductive property of fuzzy reasoning method (or algorithm) is completely satisfied. This means that the given consequent vector(resp. the given antecedent vector) is equal to fuzzy reasoning result vector, that is, $b_k^* = b_k, k = 1, 2, \cdots, r$, i.e., $B^* = B$, (resp. $a_k^* = a_k, k = 1, 2, \cdots, r$, i.e., $A^* = A$), for FMP (resp. FMT). In other words, the larger $RPCF_{FMP-FR}$(resp. $RPCF_{FMT-FR}$)is, the more the result of fuzzy reasoning satisfies the reductive property, and the smaller $RPCF_{FMP-FR}$(resp. $RPCF_{FMT-FR}$)is, the less it satisfies. At worst, when criterion function $RPCF_{FMP-FR} = 0(\%)$ and $RPCF_{FMT-FR} = 0(\%)$ , then the fuzzy reasoning method does not completely satisfy. Therefore the reductive property criterion function about every fuzzy reasoning method in the fuzzy systems satisfies $0 \leq RPCF_{FMP-FR} \leq 100$, $0 \leq RPCF_{FMT-FR} \leq 100$ for FMP and FMT, respectively. These definitions differ largely from the several previous ones [3,5,10]. Therefore according to our definition method the fuzzy reasoning result can be more correctly evaluated, and effectively used in a lot of the practical problems.

## 3.3. New Fuzzy Reasoning Method For FMP

In this subsection we define several concepts and formulate the new FMP-MD method based on moving distance. This paper's idea for FMP is as follows. In this paper we first regards that the given premise $A^*$ is moved from the antecedent $A$ in fuzzy rule. Thereby we consider a moving distance. This moving distance is the difference of the membership functions between the antecedent $A$ and the given premise $A^*$. Therefore this moving distance is fuzzy one. Next this moving distance is reflected in the consequent part of the fuzzy rule, and new conclusion of fuzzy approximate reasoning is obtained. From this consideration our new idea means that fuzzy moving distance from the antecedent $A$ to the given premise $A^*$ is equal to the fuzzy moving distance from the consequent $B$ to new conclusion $B^*$. The method presented in [15] is based on moving operation on horizontal axis, whereas our proposed method on moving operation on vertical axis, respectively, therefore [15]'s distance is usually crisp one, and this paper's distance is fuzzy one. Here moving distance is based on Euclidian distance measure. Other distance measure can be used in fuzzy approximate reasoning. According to the paper [36], general distance is as follows. Let $F_0(R)$ be all continuous fuzzy subsets of $R$ whose $\alpha$-cuts are always bounded intervals. These will be called fuzzy numbers and are the fuzzy sets most widely used in practical applications. We need to be able to compute the distance between any fuzzy set $A$ and $B$ in $F_0(R)$. We know how to find the distance between two real numbers $x, y$. The distance is $|x - y| = DM(x, y)$. We also know how to find the distance between two points in $R^2$. The function $DM(x, y)$used to compute distance is called a distance measure (DM). The basic properties of DM, i.e., $DM(x, y)$ for every $x, y$ in real space $R$ are:

- $DM(x, y) \geq 0$; i.e., distance is not negative;
- $DM(x, y) = DM(y, x)$; i.e., distance is symmetric;
- $DM(x, y) = 0$; if and only if $x = y$; i.e., we get zero distance only when $x = y$
- $DM(x, y) \leq DM(x, z) + DM(z, y)$; i.e., it is shorter to go directly from $x$ to $y$ instead of first going to intermediate point $z$.

**Definition 3.8** *Let the antecedent $A$ and the given premises $A_l^*$ for FMP be their discrete vector $A = [a_1, a_2, \cdots, a_k, \cdots, a_r], A_l^* = [a_{1l}^*, a_{2l}^*, \cdots, a_{kl}^*, \cdots, a_{rl}^*]$, respectively, where $a_k, a_{kl}^*$ are individual element of $A, A_l^*$, which are membership values in its fuzzy set, respectively. For FMP the individual elements $\alpha_{kl}$ of difference vector $\alpha_l = [\alpha_{1l}, \alpha_{2l}, \cdots, \alpha_{kl}, \cdots, \alpha_{rl}], (k = 1, 2, \cdots, r, l = 1, 2, \cdots, S)$, are defined as follows.*

$$\alpha_{kl} = a_{kl}^* - a_k, \text{for FMP} \tag{16}$$

**Definition 3.9** *Let a discrete sign vector be $p_l = [p_{1l}, p_{2l}, \cdots, p_{kl}, \cdots, p_{rl}]$, $(l = 1, 2, \cdots, s)$. Then element $p_{kl}$ of the sign vector is defined by two ways, i.e., $P(+1, 0, -1)$ form and $P(+1, -1)$ form, for FMP, as following formula, respectively.*

$$P(+1, 0, -1)form; P_{kl} = sign(\alpha_{kl}) = \begin{cases} +1, & \alpha_{kl} > 0 \\ 0, & \alpha_{kl} = 0 \\ -1, & \alpha_{kl} < 0 \end{cases} \tag{17}$$

$$P(+1,-1)form; P_{kl} = sign(\alpha_{kl}) = \begin{cases} +1, & \alpha_{kl} \geq 0 \\ -1, & \alpha_{kl} < 0 \end{cases} \tag{18}$$

**Definition 3.10** (See [31])*For FMP, the moving distance MD between the antecedent fuzzy set $A$ and the given premise $A_l^*$ by using Euclidian distance measure is defined as follows.*

$$MD(A_l^*, A) = [\sum_{k=1}^{r}[a_{kl}^* - a_k]^2/r]^{1/2}, \text{ for FMP} \tag{19}$$

**Definition 3.11** *The quasi-fuzzy reasoning result $\tilde{B}$ for FMP can be defined as follows.*

$$\tilde{B}_l = \begin{cases} B + MD(A_l^*, A) \times P_l, & if \quad \text{Case 1, 2, and 3} \\ 1 - B + MD(A_l^*, A) \times P_l, & if \quad \text{Case 4} \\ s.t.\ B + MD(A_l^*, A) \times P_l, & if \quad \text{Case 5} \end{cases} \tag{20}$$

**Definition 3.12** *The maximum $\xi_l$ and minimum $\eta_l$ of the quasi-fuzzy reasoning result $\tilde{B}_l$ are defined as follows, respectively.*

$$\xi_l = \max_{1\leq k\leq r} \tilde{B}_l, \eta_l = \min_{1\leq k\leq r} \tilde{B}_l, \quad \text{for FMP} \tag{21}$$

**Definition 3.13** *The fuzzy reasoning conclusion result for solving fuzzy modus ponens problem based on MD can be defined as formula (22), in this paper.*

$$B_l^* = \begin{cases} (\tilde{B}_l - \eta_l)/(\xi_l - \eta_l), & A_l^* \cap A \neq \Phi \\ 0, & A_l^* \cap A = \Phi \end{cases} \tag{22}$$

Where $l = 1, 2, \cdots, s$ is index of the given premises $A_l^*$ for FMP, that is, $B_l^*$ is fuzzy reasoning conclusion by the $l^{th}$ given premise $A_l^*$ for FMP. And $\Phi \in F(X)$ is an empty set, $X$ is universe of discourse, and $x \in X, A \in F(X)$. Here, $A_l^*, \tilde{A}_l$ and $A$ are the fuzzy sets in $F(X)$. The formula (22) is an standardization expression of the quasi-fuzzy reasoning result $\tilde{B}_l$ for FMP. The proposed method expressed by formula (22) is called moving distance method for the FMP with single input single output fuzzy system in this paper, for short FMP-MD. When combined $B_l^*$ and $A_l^*$, the fuzzy reasoning conclusion $B^*$ for FMP-MD can be described as follows.

$$B^* = \bigcup_{l=1}^{S} B_l^*, \quad \text{for FMP-MD} \tag{23}$$

Where $\cup$ is not max, but means the union of individual fuzzy sets obtained by fuzzy reasoning for FMP. Consequently, as defined in subsection 3.2, the criterion function $RPCF_{FR}$ for checking of the reductive property of fuzzy reasoning method is reflecting the degree of consistency between consequent $B$ and conclusion $B^*$ by formula (23), which is based on the degree of consistency between the antecedent $A$ and the given premise $A^*$ for FMP. Therefore it can be reasonable to consider the degree of consistency between conclusion $B^*$ and consequent $B$ to evaluate the reductive property (or reducibility) of FMP with considering the degree of consistency between the antecedent $A$ and the given premise $A^*$. For classical 2-valued logic, general modus ponens may be interpreted as if *"if $x$ is $A$ then $y$ is $B$"* and "$A^* = A$" then "$B^* = B$". According to fuzzy logic, we hope to provide logical analysis for fuzzy modus ponens. Based on moving distance, FMP solution can be interpreted as if *"if $x$ is $A$ then $y$ is $B$"* and *"$A^*$ is* *closer to $A$"* then "$B^*$ *is closer to* $B$". From the logical analysis of FMP solution, we can find that the conclusion $B^*$ not only relates to $A^*$ and "*$x$ is $A$ then $y$ is $B$*", but also relates to the moving distance of $A^*$ and $A$. How to select $MD(B^*, B)$ to make the conclusion of fuzzy reasoning more reasonable? We hope that $MD(B^*, B)$ is equal to $MD(A^*, A)$. And this property is proper with respect to fuzzy reasoning. Our aim is to search the fuzzy sets $B^*$ such that the moving distance $DM(B^*, B)$ should be fully supported by moving distance $MD(A^*, A)$. That is, following formula should be satisfied.

$$MD(B^*, B) = MD(A^*, A), \quad \text{for FMP} \tag{24}$$

There are a lot of fuzzy subsets on $Y$ that satisfy the formula (1). We hope the fuzzy subset as the conclusion of fuzzy reasoning satisfying the reductive property to be selected as correctly as possible.

***Principle for solving of FMP-MD Problem.*** *The FMP-MD conclusion $B^*$ of the formula (1) for a moving distance is the fuzzy subset of Y satisfying the formula (24)*

According to this principle, FMP-MD method is as follows.

**Theorem 3..1.** *Assume that moving distance is Euclidean metric, then the FMP-MD solution of the formula (1) satisfying the formula (24) is described as follows.*

$$B^* = \begin{cases} f(B + MD(B^*, B)), & if \quad \textit{Case 1, 2, and 3} \\ f(1 - B + MD(B^*, B)), & if \quad \textit{Case 4} \\ f(s.t.\ B + MD(B^*, B)), & if \quad \textit{Case 5} \end{cases} \tag{25}$$

*where $f$ is standardization operator. Therefore there is no information loss of the fuzzy reasoning processing by $f$ in SISO fuzzy system for FMP-MD.*

*Proof.* (i) Let's consider for Case 1, Case 2, and Case 3. For FMP-MD it is evident that if $A_l^* \cap A = \Phi$ then $B^* = 0$. When $A_l^* \cap A \neq \Phi$ then the fuzzy reasoning conclusion for FMP is obtained as follows.

$$
\begin{aligned}
B^* &= \bigcup_{l=1}^{S} B_l^* = B_1^* \cup B_2^* \cup \cdots B_l^* \cup B_S^* \\
&= (\tilde{B_1} - \eta_1)/(\xi_1 - \eta_1) \cup (\tilde{B_2} - \eta_2)/(\xi_2 - \eta_2) \cup \cdots \cup (\tilde{B_l} - \eta_l)/(\xi_l - \eta_l) \\
&\quad \cup \cdots \cup (\tilde{B_S} - \eta_S)/(\xi_S - \eta_S) \\
&= (B_1 + MD(A_1^*, A) \times P_1 - \eta_1)/(\xi_1 - \eta_1) \cup (B_2 + MD(A_2^*, A) \times P_2 - \\
&\quad \eta_2)/(\xi_2 - \eta_2) \cup \cdots \cup (B_l + MD(A_l^*, A) \times P_l - \eta_l)/(\xi_l - \eta_l) \cup \cdots \\
&\quad \cup (B_S + MD(A_S^*, A) \times P_S - \eta_S)/(\xi_S - \eta_S) \\
&= (B_1 \cup B_2 \cup \cdots \cup B_l \cup \cdots B_s) + (MD(A_1^*, A) \times P_1 - \eta_1)/(\xi_1 - \eta_1) \cup \\
&\quad (MD(A_2^*, A) \times P_2 - \eta_2)/(\xi_2 - \eta_2) \cup \cdots \cup (MD(A_l^*, A) \times P_l - \eta_l)/ \\
&\quad (\xi_l - \eta_l) \cup \cdots \cup (MD(A_S^*, A) \times P_S - \eta_S)/(\xi_S - \eta_S) \\
&= \bigcup_{l=1}^{S} B_l + \bigcup_{l=1}^{S} (MD(A_l^*, A) \times P_l - \eta_l)/(\xi_l - \eta_l) \\
&= \bigcup_{l=1}^{S} B_l + \bigcup_{l=1}^{S} f(MD(A_l^*, A)) \\
&= \bigcup_{l=1}^{S} B_l + f(\bigcup_{l=1}^{S} MD(A_l^*, A)) \\
&= f(B + MD(A^*, A)) \\
&= f(B + MD(B^*, B))
\end{aligned}
$$

(ii) Let us consider for Case 4. The proof of (ii) is similar to (i), so it is abbreviated.

(iii) Let us consider for Case 5. The proof of (iii) is also similar to (i), thus it is also abbreviated.

Thus we have proved that fuzzy reasoning conclusion $B^*$ for FMP-MD obtained by the formula (25) satisfies the formula (24). The information loss is guaranteed by maximum $\xi_l$ and minimum $\eta_l$ of quasi-reasoning result $\tilde{B_l}$ in above formula for FMP. □

## 3.4. New Fuzzy Reasoning Method For FMT

This paper's idea for FMT is similar to FMP, because FMT is opposite to FMP, therefore it is omitted here.

In this subsection we define several concepts and formulate new FMT-MD method based on DM presented in [33].

**Definition 3.14** *Let the antecedent* $B$ *and the given premises* $B_l^*$ *for FMT be their discrete fuzzy vector* $B = [b_1, b_2, \cdots, b_k, \cdots, b_r]$, *and* $B_l^* = [b_{1l}^*, b_{2l}^*, \cdots, b_{kl}^*, \cdots, b_{rl}^*]$, $(k = 1, 2, \cdots, r)$, *respectively. Where* $b_k$, *and* $b_{kl}^*$ *are individual elements of* $B$ *and* $B_l^*$, *which are membership values in its fuzzy set, respectively. For FMT the individual elements* $\beta_{kl}$ *of difference vector* $\beta_l = [\beta_{1l}, \beta_{2l}, \cdots, \beta_{kl}, \cdots, \beta_{rl}]$ *are defined as follows.*

$$\beta_{kl} = b_{kl}^* - b_k, \quad \text{for FMT} \tag{26}$$

**Definition 3.15** *Let a discrete sign vector be* $p_l = [p_{1l}, p_{2l}, \cdots, p_{kl}, \cdots, p_{rl}]$*,*$(l = 1, 2, \cdots, s)$*. Then element* $p_{kl}$ *of the sign vector is defined by two ways, i.e.,* $P(+1, 0, -1)$ *form and* $P(+1, -1)$ *form, for FMT, as following formulas, respectively.*

$$P(+1,0,-1)form; P_{kl} = sign(\beta_{kl}) = \begin{cases} +1, & \beta_{kl} > 0 \\ 0, & \beta_{kl} = 0 \\ -1, & \beta_{kl} < 0 \end{cases} \quad (27)$$

$$P(+1,-1)form; P_{kl} = sign(\beta_{kl}) = \begin{cases} +1, & \beta_{kl} \geq 0 \\ -1, & \beta_{kl} < 0 \end{cases} \quad (28)$$

**Definition 3.16** (See [31]) For FMT, the moving distance $MD(B_l^*, B)$ between the antecedent fuzzy set $B$ and the given premise $B_l^*$ by using Euclidian moving distance is defined as follows, according to the paper [21] .

$$MD(B_l^*, B) = [\sum_{k=1}^{r} [b_{kl}^* - b_k]^2 / r]^{1/2}, \quad \text{for FMT} \quad (29)$$

**Definition 3.17** *The quasi-fuzzy reasoning result* $\tilde{A}_l$ *for FMT can be defined as follows.*

$$\tilde{A}_l = \begin{cases} 1 - A + MD(B_l^*, B) \times P_l, & if \quad \text{Case 6, 7, and 8} \\ A + MD(B_l^*, B) \times P_l, & if \quad \text{Case 9} \\ s.t.\ A + MD(B_l^*, B) \times P_l, & if \quad \text{Case 10} \end{cases} \quad (30)$$

**Definition 3.18** *The maximum* $\xi_l$ *and minimum* $\eta_l$ *of the quasi-fuzzy reasoning result* $\tilde{A}_l$, $(l = 1, 2, \cdots, s)$ *are defined as follows, respectively.*

$$\xi_l = \max_{1 \leq k \leq r} \tilde{A}_l, \qquad \eta_l = \min_{1 \leq k \leq r} \tilde{A}_l, \quad \text{for FMT} \quad (31)$$

**Definition 3.19** *The fuzzy reasoning results for solving of fuzzy modus tollens based on MD is defined as formula (32).*

$$A_l^* = \begin{cases} (\tilde{A}_l - \eta_l)/(\xi_l - \eta_l), & B_l^* \cap B \neq \Phi \\ 0, & B_l^* \cap B = \Phi \end{cases} \quad (32)$$

Where $l = 1, 2, \cdots, s$ is index of the given premises $B_l^*$ for FMT, that is, $A_l^*$ is fuzzy reasoning conclusion by the $l^{th}$ given premise $B_l^*$ for FMT. And $\Phi \in F(Y)$ is an empty set for FMT, also $Y$ is universe of discourse, and $y \in Y, B \in F(Y)$. Here $B_l^*, \tilde{B}_l$, and $B$ are the fuzzy sets in $F(Y)$. The formula (32) is a standardization expression of the quasi-fuzzy reasoning result $\tilde{A}_l$ for FMT. The proposed method expressed by formula (32) is called moving distance method of fuzzy reasoning for FMT with single input single output fuzzy system in this paper, for short FMT-MD. When combined $B_l^*$ and $A_l^*$, the fuzzy reasoning conclusion $A^*$ for FMT can be described as follows.

$$A^* = \bigcup_{l=1}^{S} A_l^*, \quad \text{for FMT} \quad (33)$$

Where ∪ is not max, which means the union of individual fuzzy sets obtained by fuzzy reasoning for FMT. Consequently, as defined in subsection 3.1, the criterion function $RPCF_{FR}$ for checking of the reductive property of fuzzy reasoning method is reflecting the degree of consistency between the consequent $A$ and the conclusion $A^*$ by the formula (32), which is based on the degree of consistency between the antecedent $B$ and the given premise $B^*$ for FMT. Therefore it can be reasonable to consider the degree of consistency between the fuzzy reasoning conclusion $A^*$ and consequent $A$ and the given premise to evaluate the reductive property (or reducibility) of FMT with considering the consistency between the antecedent $B^*$ . For classical 2-valued logic, general modus ponens may be interpreted as if *"if $x$ is $A$ then $y$ is $B$"* and $A^* = A$ then $B^* = B$. According to fuzzy logic, we hope to provide logical analysis for fuzzy modus ponens. Based on moving distance, FMP solution can be interpreted as if *"if $x$ is $A$ then $y$ is $B$"* and *"$A^*$ is closer to A"* then *"$B^*$ is closer to B"*. From the logical analysis of FMP solution, we can find that the conclusion $B^*$ not only relates to $A^*$ and *"$x$ is $A$ they $y$ is $B$"*, but also relates to the moving distance of $A^*$ and $A$. How to select $MD(B^*, \overline{B})$ to make the conclusion of fuzzy reasoning more reasonable? We hope that $MD(B^*, \overline{B})$ is equal to $MD(A^*, \overline{A})$. And this property is proper with respect to fuzzy reasoning. Our aim is to search the fuzzy sets $B^*$ such that the moving distance $MD(B^*, \overline{B})$ should be fully supported by moving distance $MD(A^*, \overline{A})$. Let us consider FMT-MD. Here our aim is to search the fuzzy sets $A^*$ such that the moving distance $MD(A^*, \overline{A})$ obtained by the fuzzy reasoning conclusion and the consequent should be fully supported by moving distance $MD(B^*, \overline{B})$ obtained by the given premise and the antecedent. That is, following formula for FMT should be satisfied.

$$MD(A^*, \overline{A}) = MD(B^*, \overline{B}), \quad \text{for FMT} \tag{34}$$

There are a lot of fuzzy subsets on $X$ that satisfy the formula (3). We try to select the fuzzy subset as the conclusion of fuzzy reasoning satisfying the formula (34).

***Principle for solving of FMT-MD Problem.*** *The proposed FMT-MD conclusion $A^*$ of the formula (3) for a moving distance is the fuzzy subset of X satisfying the formula (34).* According to this principle, the proposed FMT-MD method is as follows.

**Theorem 3..2.** *Assume that moving distance is Euclidean metric, then the FMT-MD solution of the formula (3) satisfying the formula (34) is expressed as follows. Where $f$ is standardization operator. Hereby there is no information loss of the fuzzy reasoning processing by mapping $f$ in SISO fuzzy system for the FMT-MD.*

$$A^* = \begin{cases} f(\overline{A} + MD(A^*, A)), & if \quad \text{Case 6, 7, and 8} \\ f(A + MD(A^*, A)), & if \quad \text{Case 9} \\ f(s.t.\ A + MD(A^*, A)), & if \quad \text{Case 10} \end{cases} \tag{35}$$

As known from the formula (25) and (35), FMT is opposite to FMP. So its proof is omitted here.

# 4. Checking of CRI, TIP, AARS, QIP, and Proposed MDM

In this section we consider validation of the new fuzzy approximate reasoning method based on fuzzy moving distance through several example and MATLAB experiments. First let us discuss checking for the reductive property of fuzzy reasoning method.

## 4.1. Several Examples

**Example 4..1.** *Assume that the fuzzy sets of the rule are given as* $A(x) = [1, 0.3, 0, 0, 0]$, $B(y) = [0, 0, 0, 0.3, 1]$*, and the given premise for FMP* $A^*(x) = A(x) = [1, 0.3, 0, 0, 0]$*, the premise for FMT* $B^*(y) = B(y) = [0, 0, 0, 0.3, 1]$*, then the new conclusion reasoning result by any fuzzy reasoning method (for instance WW) is obtained as* $B^*(y) \neq B(y) = [0, 0, 0.1, 0.4, 1]$ *for FMP,* $A^*(x) \neq A(x) = [1, 0.7, 0.4, 0.1, 0]$ *for FMT, respectively. At this time according to our new method, the criterion function is calculated as* $RPCF_{WW} = \frac{1}{2}(RPCF_{FMP-WW-I} + RPCF_{FMT-WW-I}) = \frac{1}{2}(96 + 38) = 67.00(\%)$*. Consequently the reductive property of a fuzzy reasoning method WW is satisfied as 67.00(%). But according to[2], since* $B^*(y) \neq B(y)$ *for FMP and* $A^*(x) \neq A(x)$ *for FMT, the reductive property of a fuzzy reasoning method WW is not satisfied as 0(%), thus their evaluation is strict and not right.*

**Example 4..2.** *Let us consider the reductive property of Example 1 and 2 in [7] . The fuzzy sets of the rule are as* $[small] = [1, 0.3, 0, 0, 0]$, $[large] = [0, 0, 0, 0.3, 1]$ *and the premise* $[medium] = [0, 0.3, 1, 0.3, 0]$*. The conclusions by CRI are obtained as* $B^*(y) = [1, 1, 1, 1, 1]$ *for FMP, and* $A^*(x) = [0.3, 0.7, 1, 1, 1]$ *for FMT. According to our Definition 3.3, 3.6, and 3.7, the reductive property of a fuzzy reasoning method WW is as follows.*

$RPCF_{WW} = \frac{1}{2}(RPCF_{FMP-WW-I} + RPCF_{FMT-WW-I}) = \frac{1}{2}(43.27 + 89.5) = 66.38(\%)$

Now let us consider the difference of ours and [20]'s checking method. [20] 's checking method is very strict and has some weakness. [19]'s weakness is as follows. For the same Case such as Example 4.1, according to [30]'s checking method, the fuzzy approximate reasoning result are $B^*(y) = [0, 0, 0.1, 0.4, 1] \neq B(y) = [0, 0, 0, 0.3, 1]$ for FMP, and $A^*(x) = [1, 0.7, 0.4, 0.1, 0] \neq A(x) = [1, 0.3, 0, 0, 0]$ for FMT, respectively. Then the reductive property of the fuzzy reasoning method WW is not satisfied by the criterion of Table 1, that is, it is not flexible and soft. For this evaluation, considering by [17]'s viewpoint, it is satisfied as 0(%) or is not satisfied as 100(%), vice versa. So in order to overcome [17]'s weakness, we generalized and extended the criterion for FMP and FMT shown in Table 1 according to [20]. Frankly speaking, even though reasoning results are obtained as $B^*(y) \neq B(y)$ for FMP, and $A^*(x) \neq A(x)$ for FMT, respectively, our checking method can discuss the degree of coincidence between the given premise and the antecedent of fuzzy rule. In other words our proposed criterion function (15) tries to calculate the percentage degree of coincidence between the consequent $B(y)$ (resp. $A(x)$ of fuzzy rule and the conclusion $B^*(y)$(resp. $A^*(x)$) of the reasoning, and then calculate the average of two percentage degrees of coincidence for FMP and FMT. The higher the degree of coincidence between $B(y)$ (resp. $A(x)$ and $B^*(y)$ (resp. $A^*(x)$ ) is, the better the reductive property of FMP (resp. FMT) is. Therefore as shown in Example 4.1 and 4.2, our new checking method of the reductive property is softer and better in accordance with general human understanding and practical problems than [20]'s one. Below two examples for proposed FMP-MD are shown.

**Example 4..3.** *Let us consider for the FMP-MD in Class 1. According to moving distance for FMP, FMP-MD can be obtained as the formula (22). For the antecedent* $A = [1, 0.3, 0, 0, 0]$*, the consequent* $B = [0, 0, 0, 0.3, 1]$*, for four Cases, fuzzy reasoning results are as follows. In Case 1, the given premise is* $A^* = A$*, moving distance* $DM$ *is calculated as* $DM = 0$*, quasi-reasoning result* $\tilde{B}$ *for FMP-MD is calculated as* $\tilde{B} = [0, 0, 0, 0.3, 1]$*, since* $B^* = \tilde{B} = [0, 0, 0, 0.3, 1] = B$*, therefore the reductive property is 100(%). In Case 2, 3, and 4, the reductive property is* *calculated as 91.16(%), 92.83 (%), and 68.25(%), respectively. Thus total reductive property criterion function value for FMP-MD presented in this paper is obtained as* $RPCF_{FMP-DM} = 87.729(\%)$.

**Example 4..4.** *Let us consider another example for the FMP-MD in Class 1. For example, let the fuzzy rule is "If $x$ is $A$ Then $y$ is $B$", consequent $B = [0.2, 0.4, 0.5, 0.7, 1]$, the quasi-reasoning result $\beta = B, \xi = 1$ and $\eta = 0.2$ when the premise is "$x$ is $A$". It is obvious that the reasoning result $B^*$ is not equal to $B$. A lot of detailed calculations are carried out in this paper, but full text entirely based on two examples in reference paper.*

## 4.2. Several Experimental Results by MATLAB

In this subsection we chek the reductive property of CRI, TIP, AARS, QIP, and proposed MDM by MATLAB experiment. The most general forms of the CRI solutions of FMP and FMT are as the formula (36) and (37).

$$\text{CRI-FMP;}\quad B^*(y) = \bigvee_{x\in U} (A^*(x) \otimes (A(x) \to B(y))) \tag{36}$$

$$\text{CRI-FMT;}\quad A^*(x) = \bigvee_{y\in V} (B^*(y) \otimes (A(x) \to B(y))) \tag{37}$$

FMP-CRI and FMT-CRI reductive properties based on the formula (36) and (37) for Class 1 are shown in Table 2.

The reductive property of FMP-CRI and by Łukasiewicz, Gödel, $R_0$ and Gougen are more than FMT-CRI with respect to [3], respectively. The general forms of the TIP are as follows. [37]

$$\text{TIP-FMP;}\quad B^*(y) = \bigvee_{x\in U} (A^*(x) \otimes (A(x) \to B(y))) \tag{38}$$

$$\text{TIP-FMT;}\quad A^*(x) = \bigwedge_{y\in V} ((A^*(x) \to (B(y)) \to B^*(y)) \tag{39}$$

The experimental result about the reductive property for FMP-TIP and FMT-TIP in Class 1 is shown in Table 3. The reductive property of FMT-TIP by Łukasiewicz, Gödel, $R_0$ and Gougen are more than FMP-TIP with respect to [3], respectively. Next, we check the reductive property of Approximate Analogical Reasoning Schema (AARS). Unlike CRI [43], in [31, 32], a similarity-based fuzzy reasoning method, i.e., Turksen and Zhong's AARS was proposed. The AARS modifies the consequent based on the similarity (closeness) between the given premise $A^*$ and the antecedent $A$.

If the degree of similarity measure is greater than the predefined threshold value, then the rule will be fired and the consequent is deduced by some modification techniques. In [32], distances measure (DM) for FMP (resp. FMT) is as follows.

$$DM = D_2(A^*, A) = [\sum_{i=1}^{n} [\mu_{A^*}(x_i) - \mu_A(x_i)]^2/n]^{1/2} \tag{40}$$

$$DM = D_2(B^*, B) = [\sum_{i=1}^{n} [\mu_{B^*}(y_i) - \mu_B(y_i)]^2/n]^{1/2} \tag{41}$$

**Table 2.** FMP-CRI and FMT-CRI reductive property in Class 1

| FMP-CRI Premise $A^*(x)$ | FMP-CRI-Conclusion $B^*(y)$ and Reductive Property | | | |
|---|---|---|---|---|
| | FMP- CRI -Łukasiewicz | | FMP- CRI -Gödel | |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 100.00 % | [0, 0, 0, 0.3, 1] | 100.00% |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.80 % | [0, 0, 0, 0.3, 1] | 95.80% |
| [1, 0.548, 0, 0, 0] | [0.248, 0.248, 0.248, 0.548, 1] | 85.14 % | [0, 0, 0, 0.548, 1] | 100.00% |
| [0, 0.7, 1, 1, 1] | [1, 1, 1, 1, 1] | 74.00 % | [1, 1, 1, 1, 1] | 74.00 % |
| $RPCF_{FMP-CRI-I}$ | 88.73% | | 92.45% | |
| FMT-CRI Premise $B^*(y)$ | FMT-CRI-Conclusion $A^*(x)$ and Reductive Property | | | |
| | FMT- CRI -Łukasiewicz | | FMT- CRI -Gödel | |
| [1, 1, 1, 0.7, 0] | [1, 1, 1, 1, 1] | 74.00 % | [1, 1, 1, 1, 1] | 74.00 % |
| [1, 1, 1, 0.91, 0] | [1, 1, 1, 1, 1] | 78.20 % | [1, 1, 1, 1, 1] | 78.20% |
| [1, 1, 1, 0.452, 0] | [1, 1, 1, 1, 1] | 69.05 % | [1, 1, 1, 1, 1] | 69.05 % |
| [0, 0, 0, 0.3, 1] | [1, 1, 1, 1, 1] | 26.00 % | [1, 1, 1, 1, 1] | 26.00 % |
| $RPCF_{FMT-CRI-I}$ | 61.81% | | 61.81% | |
| FMP-CRI Premise $A^*(x)$ | FMP-CRI-Conclusion $B^*(y)$ and Reductive Property | | | |
| | FMP- CRI -$R_0$ | | FMP- CRI -Gougen | |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 100.00 % | [0, 0, 0, 0.3, 1] | 100.00 % |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.80 % | [0, 0, 0, 0.3, 1] | 95.80% |
| [1, 0.548, 0, 0, 0] | [0.548,0.548, 0.548,0.548,1] | 67.14 % | [0, 0, 0, 0.548,1] | 100.00% |
| [0, 0.7, 1, 1, 1] | [1, 1, 1, 1, 1] | 74.00 % | [1, 1, 1, 1, 1] | 74.00% |
| $RPCF_{FMP-CRI-I}$ | 84.23% | | 92.45% | |
| FMT-CRI Premise $B^*(y)$ | FMT-CRI-Conclusion $A^*(x)$ and Reductive Property | | | |
| | FMT- CRI -$R_0$ | | FMT- CRI -Gougen | |
| [1, 1, 1, 0.7, 0] | [1, 1, 1, 1, 1] | 74.00 % | [1, 1, 1, 1, 1] | 74.00 % |
| [1, 1, 1, 0.91, 0] | [1, 1, 1, 1, 1] | 78.20 % | [1, 1, 1, 1, 1] | 78.20 % |
| [1, 1, 1, 0.452, 0] | [1, 1, 1, 1, 1] | 69.05 % | [1, 1, 1, 1, 1] | 69.05% |
| [0, 0, 0, 0.3, 1] | [1, 1, 1, 1, 1] | 26.00 % | [1, 1, 1, 1, 1] | 26.00 % |
| $RPCF_{FMT-CRI-I}$ | 61.81% | | 61.81% | |

The similarity by DM is then defined as follows.

$$S_{AARS} = (1 + DM)^{-1} \tag{42}$$

If the rule is fired, then the consequent is modified by a modification function which could appear in one of the two forms for FMP and FMT, i.e., more or less form and, fuzzy membership value reduction form, for short, reduction form, according to [32], respectively.

**Table 3.** FMP-TIP and FMT-TIP reductive property in Class 1

| FMP-TIP Premise $A^*(x)$ | FMP-TIP-Conclusion $B^*(y)$ and Reductive Property | |
|---|---|---|
| | FMP-TIP-Łukasiewicz | FMP-TIP-Gödel |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] 100.00 % | [0, 0, 0, 0.3, 1] 100.00 % |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] 95.80 % | [0, 0, 0, 0.3, 1] 95.80 % |
| [1, 0.548, 0, 0, 0] | [0.248, 0.248, 0.248, 0.548, 1] 85.14 % | [0, 0, 0, 0.548, 1] 100.00 % |
| [0, 0.7, 1, 1, 1] | [1, 1, 1, 1, 1] 74.00 % | [1, 1, 1, 1, 1] 74.00 % |
| $RPCF_{FMP-TIP-I}$ | 88.73 % | 92.45 % |
| FMT-TIP Premise $B^*(y)$ | FMT-TIP-Conclusion $A^*(x)$ and Reductive Property | |
| | FMT-TIP-Łukasiewicz | FMT-TIP-Gödel |
| [1, 1, 1, 0.7, 0] | [0, 0, 0, 0, 0] 26.00 % | [0, 0, 0, 0, 0] 26.00 % |
| [1, 1, 1, 0.91, 0] | [0, 0, 0, 0, 0] 21.80 % | [0, 0, 0, 0, 0] 21.80 % |
| [1, 1, 1, 0.452, 0] | [0, 0, 0, 0, 0] 30.95 % | [0, 0, 0, 0, 0] 30.95 % |
| [0, 0, 0, 0.3, 1] | [1, 0.3, 0, 0, 0] 100.00 % | [1, 0.3, 0, 0, 0] 100.00 % |
| $RPCF_{FMT-TIP-I}$ | 44.69 % | 44.69 % |
| FMP-TIP Premise $A^*(x)$ | FMP-TIP-Conclusion $B^*(y)$ and Reductive Property | |
| | FMP-TIP-$R_0$ | FMP-TIP-Gougen |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] 100.00 % | [0, 0, 0, 0.3, 1] 100.00 % |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] 95.80 % | [0, 0, 0, 0.3, 1] 95.80 % |
| [1, 0.548, 0, 0, 0] | [0.548, 0.548, 0.548, 0.548, 1] 67.14 % | [0, 0, 0, 0.548, 1] 100.00% |
| [0, 0.7, 1, 1, 1] | [1, 1, 1, 1, 1] 74.00 % | [1, 1, 1, 1, 1] 74.00 % |
| $RPCF_{FMP-TIP-I}$ | 84.23 % | 92.45 % |
| FMT-TIP Premise $A^*(x)$ | FMT-TIP-Conclusion $B^*(y)$ and Reductive Property | |
| | FMT-TIP-$R_0$ | FMT-TIP-Gougen |
| [1, 1, 1, 0.7, 0] | [0, 0, 0, 0, 0] 26.00 % | [0, 0, 0, 0, 0] 26.00 % |
| [1, 1, 1, 0.91, 0] | [0, 0, 0, 0, 0] 21.80 % | [0, 0, 0, 0, 0] 21.80 % |
| [1, 1, 1, 0.452, 0] | [0, 0, 0, 0, 0] 30.95 % | [0, 0, 0, 0, 0] 30.95 % |
| [0, 0, 0, 0.3, 1] | [1, 0.3, 0, 0, 0] 100.00 % | [1, 0.3, 0, 0, 0] 100.00 % |
| $RPCF_{FMT-TIP-I}$ | 44.69 % | 44.69 % |

$$\text{FMP-AARS-}\textit{more or less form};\, B^* = min\{1, B/S_{AARS}\} \tag{43}$$

$$\text{FMP-AARS-}\textit{more or less form};\, A^* = min\{1, A/S_{AARS}\} \tag{44}$$

$$\text{FMP-AARS-}\textit{reduction form};\, B^* = B \times S_{AARS} \tag{45}$$

$$\text{FMP-AARS-}\textit{reduction form};\, A^* = A \times S_{AARS} \tag{46}$$

The reductive properties of FMP-AARS and FMT-AARS shown in Table 4 are less than FMP-CRI and FMT-CRI with respect to [3], respectively.

Next, let us check the reductive property of FMP-QIP, FMT-QIP presented by [45]. The reductive property of FMP-QIP, FMT-QIP is shown in Table 5.

$$\text{FMP-QIP};\, B^*(y) = \bigvee_{x \in U} (A^*(x) \otimes (A^*(x) \to A(x))) \otimes (A(x) \to B(y))) \tag{47}$$

**Table 4.** FMP-AARS and FMT-AARS reductive property in Class 1

| FMP-AARS Premise $A^*(x)$ | FMP-AARS Conclusion and Reductive Property | | |
|---|---|---|---|
| | Conclusion $B^*(y)$ | | $RPCF_{FMP}$ |
| [1, 0.3, 0, 0, 0] | *more or less form* | [0, 0, 0, 0.3, 1] | 100.00 % |
| | *reduction form* | [0, 0, 0, 0.3, 1] | 100.00 % |
| [1, 0.09, 0, 0, 0] | *more or less form* | [0, 0, 0, 0.328, 1] | 95.24 % |
| | *reduction form* | [0, 0, 0, 0.274, 0.914] | 94.60 % |
| [1, 0.548, 0, 0, 0] | *more or less form* | [0, 0, 0, 0.333, 1] | 95.71 % |
| | *reduction form* | [0, 0, 0, 0.27, 0.9] | 92.45 % |
| [0, 0.7, 1, 1, 1] | *more or less form* | [0, 0, 0, 0.574, 1] | 17.47 % |
| | *reduction form* | [0, 0, 0, 0.157, 0.523] | 18.68 % |
| *RPCF-FMP* | *FMP-AARS-more or less form* | | 88.06 % |
| | *FMP-AARS-reduction form* | | 87.64 % |
| FMT-AARS Premise $B^*(y)$ | FMT-AARS-form Conclusion and Reductive Property | | |
| | Conclusion $A^*(x)$ | | $RPCF_{FMT}$ |
| [1, 1, 1, 0.7, 0] | *more or less form* | [1, 0.574, 0, 0, 0] | 17.47 % |
| | *reduction form* | [0.523, 0.157, 0, 0, 0] | 18.68 % |
| [1, 1, 1, 0.91, 0] | *more or less form* | [1, 0.581, 0, 0, 0] | 13.41 % |
| | *reduction form* | [0.517, 0.155, 0, 0, 0] | 14.57 % |
| [1, 1, 1, 0.452, 0] | *more or less form* | [1, 0.569, 0, 0, 0] | 17.66 % |
| | *reduction form* | [0.527, 0.158, 0, 0, 0] | 23.57 % |
| [0, 0, 0, 0.3, 1] | *more or less form* | [1, 0.3, 0, 0, 0] | 100.00 % |
| | *reduction form* | [1, 0.3, 0, 0, 0] | 100.00 % |
| *RPCF-FMT* | *FMT-AARS-more or less form* | | 88.06 % |
| | *FMT-AARS-reduction form* | | 92.98 % |

$$\text{FMT-QIP; } A^*(x) = \bigvee_{y \in V} (A^*(x) \otimes (A(x) \to B(y))) \otimes (B(y) \to B^*(y))) \tag{48}$$

Next, the proposed FMP-MD and FMT-MD reductive properties are shown in Table 6.

When compared with Table 2, FMP-MD and FMT-MD reductive property are more than FMP-CRI and FMT-CRI, respectively.

Next, 17 individual fuzzy reasoning methods are compared with respect to FMP and FMT in Class 1. (Table 7) From Table 7, in this paper, the reductive properties about TIP, CRI, QIP, and AARS are improved by our proposed MDM. From the experiment results, the reductive property ranking of the fuzzy reasoning methods in Class 1 are as follows; MDM, CRI, TIP, QIP, and AARS, respectively. From the experiment results, the fuzzy reasoning methods, i.e., CRI, TIP, and QIP, AARS, in Class 1 have some information losses, respectively. Next, we compare and analyze about CRI, TIP, AARS and proposed method MDM for Class 2. The reductive properties of four fuzzy reasoning methods in Class 2 are shown in Table 8.

In Table 8, the given premises are $A^* = s.t.\, A = [1, 0.2, 0, 0, 0]$, $B^* = s.t.\, B = [0, 0, 0, 0.2, 1]$. From Table 8, we can see that, the best is our proposed method, next best CRI, TIP, QIP and the lowest AARS in Class 2, this result is similar as in Class 1. Through the experiments we have obtained that proposed MDM is better in accordance with human thinking than existing fuzzy reasoning methods.

**Table 5.** The reductive property of FMP-QIP, FMT-QIP in Class 1

| FMP-QIP Premise $A^*(x)$ | FMP-QIP Conclusion and Reductive Property | | | |
|---|---|---|---|---|
| | FMP-QIP-Łukasiewicz | | FMP-QIP-Gödel | |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 100.00 % | [0, 0, 0, 0.3, 1] | 100.00 % |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.80 % | [0, 0, 0, 0.3, 1] | 95.80 % |
| [1, 0.548, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.05 % | [0, 0, 0, 0.3, 1] | 95.05 % |
| [0, 0.7, 1, 1, 1] | [0, 0, 0, 0.3, 1] | 26.00 % | [0, 0, 0, 0.3, 1] | 26.00 % |
| $RPCF_{FMP-QIP}$ | 79.21 % | | 79.21 % | |
| FMT-QIP Premise $B^*(y)$ | FMT-QIP Conclusion and Reductive Property | | | |
| | FMT-QIP-Łukasiewicz | | FMT-QIP-Gödel | |
| [1, 1, 1, 0.7, 0] | [0.3, 0.3, 0, 0, 0] | 26.00 % | [0.3, 0.3, 0, 0, 0] | 26.00 % |
| [1, 1, 1, 0.91, 0] | [0.3, 0.3, 0, 0, 0] | 21.80 % | [0.3, 0.3, 0, 0, 0] | 21.80 % |
| [1, 1, 1, 0.452, 0] | [0.3, 0.3, 0, 0, 0] | 30.95 % | [0.3, 0.3, 0, 0, 0] | 30.95 % |
| [0, 0, 0, 0.3, 1] | [1, 0.3, 0, 0, 0] | 100.00 % | [1, 0.3, 0, 0, 0] | 100.00 % |
| $RPCF_{FMT-QIP}$ | 44.69 % | | 44.69 % | |
| FMP-QIP Premise $A^*(x)$ | FMP-QIP Conclusion and Reductive Property | | | |
| | FMP-QIP-$R_0$ | | FMP-QIP-Gougen | |
| [1, 0.3, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 100.00 % | [0, 0, 0, 0.3, 1] | 100.00 % |
| [1, 0.09, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.80 % | [0, 0, 0, 0.3, 1] | 95.80 % |
| [1, 0.548, 0, 0, 0] | [0, 0, 0, 0.3, 1] | 95.05 % | [0, 0, 0, 0.3, 1] | 95.05 % |
| [0, 0.7, 1, 1, 1] | [0, 0, 0, 0.3, 1] | 26.00 % | [0, 0, 0, 0.3, 1] | 26.00 % |
| $RPCF_{FMP-QIP}$ | 79.21 % | | 79.21 % | |
| FMT-QIP Premise $B^*(y)$ | FMT-QIP Conclusion and Reductive Property | | | |
| | FMP-QIP-$R_0$ | | FMP-QIP-Gougen | |
| [1, 1, 1, 0.7, 0] | [0.3, 0.3, 0, 0, 0] | 26.00 % | [0.3, 0.3, 0, 0, 0] | 26.00 % |
| [1, 1, 1, 0.91, 0] | [0.3, 0.3, 0, 0, 0] | 21.80 % | [0.3, 0.3, 0, 0, 0] | 21.80 % |
| [1, 1, 1, 0.452, 0] | [0.3, 0.3, 0, 0, 0] | 30.95 % | [0.3, 0.3, 0, 0, 0] | 30.95 % |
| [0, 0, 0, 0.3, 1] | [1, 0.3, 0, 0, 0] | 100.00 % | [1, 0.3, 0, 0, 0] | 100.00 % |
| $RPCF_{FMT-QIP}$ | 44.69 % | | 44.69 % | |

## 4.3. This paper's Characteristics

### 4.3.1. Samenees with the [15]'s Method

- Proposed MDM and the [15]'s method do not use the compositional operation proposed by Zadeh, respectively, therefore both of them have not information loss, thereby have the higher reductive property than CRI.

- Proposed MDM and the [15]'s method do not use the compositional operation applied in TIP and QIP, respectively, therefore both of them have not information loss, thereby have the higher reductive property than TIP and QIP.

- Proposed MDM and the [15]'s method do not use the similarity measure and the modification function applied in AARS, respectively, therefore both of them have not information loss and have the higher reductive property than AARS.

- Proposed MDM does use moving distance operation, therefore it has not information loss from the formula (25) and (35), thereby has the highest reductive property among CRI, AARS, TIP and QIP.

**Table 6.** FMP-MD and FMT-MD conclusion and reductive property in Class 1

| FMP-MD Premise $A^*(x)$ | FMP-MD-form Conclusion and Reductive Property | | |
|---|---|---|---|
| | Conclusion $B^*(y)$ | | *RPCF*$_{FMP\text{-}MD}$ |
| [1, 0.3, 0, 0, 0] | $P(+1,0,-1)form$ | [0, 0, 0, 0.3, 1] | 100.00 % |
| | $P(+1,-1)form$ | [0, 0, 0, 0.3, 1] | 100.00 % |
| [1, 0.09, 0, 0, 0] | $P(+1,0,-1)form$ | [0.086, 0, 0.086, 0.36, 1] | 91.16 % |
| | $P(+1,-1)form$ | [0.158, 0, 0.158, 0.41, 1] | 87.26 % |
| [1, 0.548, 0, 0, 0] | $P(+1,0,-1)form$ | [0, 0.111, 0, 0.3, 1] | 92.83 % |
| | $P(+1,-1)form$ | [0, 0, 0, 0.3, 1] | 95.05 % |
| [0, 0.7, 1, 1, 1] | $P(+1,0,-1)form$ | [0, 0.646, 0.646, 0.752, 1] | 68.25 % |
| | $P(+1,-1)form$ | [0, 1, 1, 0.84, 0.45] | 68.25 % |
| $RPCF_{FMP-MD}$ | $FMP-MD-P(+1,0,-1)form$ | | 88.06 % |
| | $FMP-MD-P(+1,-1)form$ | | 87.64 % |
| FMT-MD Premise $B^*(y)$ | FMT-MD-form Conclusion and Reductive Property | | |
| | Conclusion $A^*(x)$ | | *RPCF*$_{FMT\text{-}MD}$ |
| [1, 1, 1, 0.7, 0] | $P(+1,0,-1)form$ | [0, 0.7, 1, 1, 1] | 100.00 % |
| | $P(+1,-1)form$ | [0, 0.7, 1, 1, 1] | 100.00 % |
| [1, 1, 1, 0.91, 0] | $P(+1,0,-1)form$ | [0 0.64 0.914 1 0.914] | 91.16 % |
| | $P(+1,-1)form$ | [0, 0.7, 1, 1, 1] | 95.80 % |
| [1, 1, 1, 0.452, 0] | $P(+1,0,-1)form$ | [0, 0.7, 1, 0.889, 1] | 92.83 % |
| | $P(+1,-1)form$ | [0, 0.7 1, 0.778, 1] | 90.61 % |
| [0, 0, 0, 0.3, 1] | $P(+1,0,-1)form$ | [0, 0.11, 0, 0.3, 1] | 68.25 % |
| | $P(+1,-1)form$ | [0.56, 0, 0, 0.13, 1] | 85.51 % |
| $RPCF_{FMT-MD}$ | $FMT-MD-P(+1,0,-1)form$ | | 88.06 % |
| | $FMT-MD-P(+1,-1)form$ | | 92.98 % |

**Table 7.** Comparisons of CRI, TIP, QIP, AARS and proposed MDM with respect to the reductive property in Class 1

| No | In Class 1 Fuzzy Reasoning Method | | *RPCF*$_{FMP\text{-}FR}$ | *RPCF*$_{FMT\text{-}FR}$ | *RPCF*$_{FR}$ |
|---|---|---|---|---|---|
| 1 | Proposed MDM | $P(+1,0,-1)form$ | 88.06% | 88.06% | 88.06% |
| | | $P(+1,-1)form$ | 87.64% | 88.67% | 88.16% |
| 2 | CRI(1975) | Gödel; G | 92.45% | 61.81% | 77.131% |
| | | Gougen; $G_0$ | 92.45% | 61.81% | 77.131% |
| | | Łukasiewicz; L | 88.73% | 61.81% | 75.273% |
| | | $R_0$ | 84.23% | 61.81% | 73.023% |
| | | Zadeh; Rz | 78.38% | 61.81% | 70.098% |
| 3 | TIP(1999) | Gödel; G | 92.45 % | 44.69 % | 68.570 % |
| | | Gougen; $G_0$ | 92.45 % | 44.69 % | 68.570 % |
| | | Łukasiewicz; L | 88.73 % | 44.69 % | 66.711 % |
| | | $R_0$ | 84.23 % | 44.69 % | 64.461 % |
| 4 | QIP(2015) | Łukasiewicz; L | 79.21 % | 44.69 % | 61.950 % |
| | | Gödel; G | 79.21 % | 44.69 % | 61.950 % |
| | | $R_0$ | 79.21 % | 44.69 % | 61.950 % |
| | | Gougen; $G_0$ | 79.21 % | 44.69 % | 61.950 % |
| 5 | AARS(1990) | *more or less form* | 76.43 % | 39.20 % | 57.818 % |
| | | *reduction form* | 77.10 % | 37.14 % | 57.121 % |

**Table 8.** Comparisons of CRI, TIP, QIP, AARS and Proposed MDM with respect to reductive property in Class 2

| No | In Class 2 Fuzzy Reasoning Method | | $RPCF_{FMP\text{-}FR}$ | $RPCF_{FMT\text{-}FR}$ | $RPCF_{FR}$ |
|---|---|---|---|---|---|
| 1 | CRI(1975) | Zadeh; Rz | 81.38% | 74.31% | 77.85% |
| | | Łukasiewicz; L | 94.73% | 74.31% | 84.52% |
| | | Gödel; G | 98.45% | 74.31% | 86.38% |
| | | $R_0$ | 90.23% | 74.31% | 82.27% |
| | | Gougen; $G_0$ | 98.45% | 74.31% | 86.38% |
| 2 | TIP(1999) | Łukasiewicz; L | 94.73% | 25.69% | 62.01% |
| | | Gödel; G | 98.45% | 25.69% | 62.08% |
| | | $R_0$ | 90.23% | 25.69% | 57.95% |
| | | Gougen; $G_0$ | 98.45% | 25.69% | 62.08% |
| 3 | QIP(2015) | Łukasiewicz; L | 97.21% | 25.69% | 61.45% |
| | | Gödel; G | 97.21% | 25.69% | 61.45% |
| | | $R_0$ | 97.21% | 25.69% | 61.45% |
| | | Gougen; $G_0$ | 97.21% | 25.69% | 61.45% |
| 4 | AARS(1990) | *more or less form* | 97.17% | 16.02% | 56.59% |
| | | *reduction form* | 96.11% | 18.38% | 57.25% |
| 5 | Proposed MDM | $P(+1, 0, -1)$ *form* | 94.92% | 96.08% | 94.92% |
| | | $P(+1, -1)$ *form* | 93.97% | 96.10% | 95.04% |

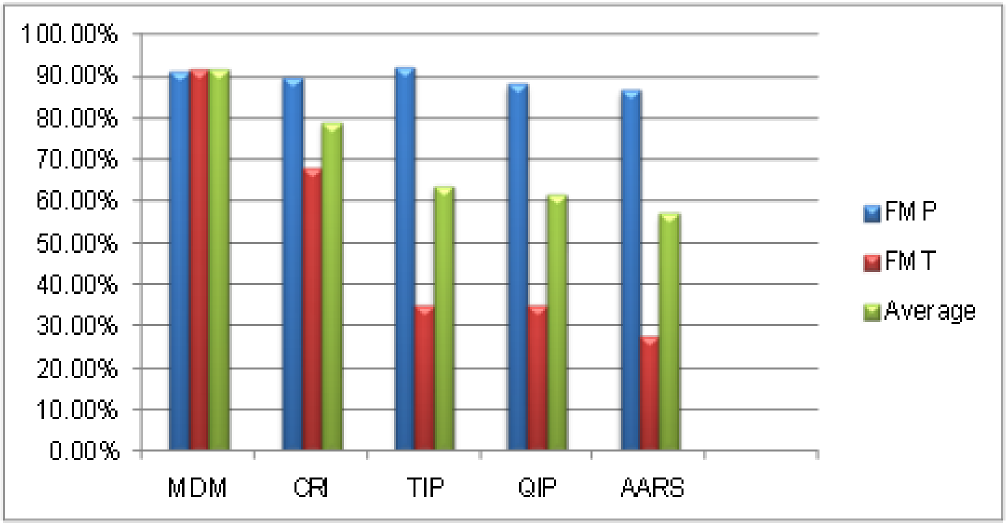
### 4.3.2. Differences with the Existing Methods

- CRI, TIP, QIP, and AARS all use nonlinear operators max, min, and several implication, for example the formula (5)-(8), therefore in fuzzy reasoning computation they have all information losses, whereas, our proposed method uses linear operators, for example, the formula (16)-(23), and the formula (26)-(32), thus proposed MDM has not information loss.
- TIP, QIP, and AARS methods are all based on Zadeh's MIN-MAX Compositional Rule of Inference (36) and (37), i.e., the formula (38)-(39), and the formula (43)-(48), and the proposed method MDM is based on a new principle (24), (34), for FMP and FMT, respectively, which means that fuzzy moving distance from the antecedent to the given premise is equal to the fuzzy moving distance from the consequent to the new conclusion.
- The method presented in [15] is based on moving distance operation on horizontal axis, whereas our proposed method on moving distance operation on vertical axis, therefore [15]'s distance is usually crisp one, and this paper's distance fuzzy one.
- The reductive property of CRI, TIP, QIP, and AARS methods for FMT are less than their FMP, whereas proposed MDM for FMT is similar to its FMP. The reason is as follows. Our MDM is considered that FMT is opposite FMP, whereas other methods are not considered like this.

## 5. Conclusions

Our original research results can be summarized as follows. We proposed reductive property criterion function for checking of the fuzzy reasoning result. And then, unlike well-known fuzzy

The total reductive properties of the 5 fuzzy reasoning methods in Class 1 and 2 are comprehensively shown in Fig. 1.




| | MDM | CRI | TIP | QIP | AARS |
|---|---|---|---|---|---|
| FMP(%) | 91.148 | 89.948 | 92.465 | 88.210 | 86.703 |
| FMT(%) | 91.938 | 68.060 | 35.190 | 35.190 | 27.685 |
| AVERAGE(%) | 91.543 | 79.004 | 63.828 | 61.700 | 57.194 |

**Fig. 1.** The comprehensive reductive properties of the 5 fuzzy reasoning methods for Class 1 and 2

reasoning methods based on the similarity measure, we have proposed a principle of new fuzzy reasoning method based on moving distance, i.e., FMP-MD and FMT-MD, for short, MDM, and then presented two theorem for FMP and FMT. The CRI, TIP, QIP, and AARS use not only linear operators but also nonlinear operators, thus they have the information loss in fuzzy reasoning. Otherwise our method uses linear operators, which has not the information loss in fuzzy reasoning, and is more than CRI, TIP, QIP and AARS with respect to the reductive property. We compared 17 individual fuzzy reasoning methods for FMP and FMT. Consequently our proposed MDM is illustratively better than AARS, TIP, QIP, and CRI with respect to the reductive property, and in accordance with human thinking.

## Acknowledgement

Authors would like to thank the editor and unknown reviewers for their helpful comments and suggestions. We are indebted to the editor and nameless reviewers for providing very useful, detailed criticism, comments, and suggestions.

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