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Languages of Fuzzy Involutive Fibonacci Words

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Abstract. Fibonacci words represent a well-known category of infinite sequences of words that have garnered significant attention in the fields of word combinatorics and theoretical computer science. Fibonacci words possess exceptional structural properties that make them stand out significantly. This study analyzes the languages produced by involutive Fibonacci words, a distinct category of Fibonacci words. Fuzzy involutive Fibonacci words are defined by incorporating membership degree into the rules of production. Their properties are examined, and the formal languages linked to these words are defined.

Key-words: Fibonacci languages; involutive Fibonacci languages; involutive Fibonacci words; membership degrees.

1. Introduction

Word combinatorics is a relationship between mathematics and computer science that focuses on combinatorial features used in the theory of computation. The study of square-free words originates from the pioneering work of Thue [1], later collected in Selected Mathematical Papers of Axel Thue [2]. This led to the exploration of combinatorics. The algebraic characteristics of combinatorics were thoroughly examined in Lothaire [3]. However, the study of formal languages is closely related to combinatorics on words, as both fields focus on the study of words.

Recently, people have been looking into different combinatorial features of words. For instance, Kari and Mahalingam looked at special types of words that have repeating parts [4]; Czeizler and others studied how basic features of words are affected by DNA activity [5]; Yu explored the repeating parts within words; and more [6]. The study then expands to encompass two-dimensional arrays and their combinatorial properties. Researchers have constructed various algorithms to study these properties. For instance, Knuth et al. [7] and Cole et al. [8] elucidated pattern matching, while Geizhals et al. [9] created an algorithm to detect maximal two-dimensional palindromes.

As a string equivalent of Fibonacci numbers, Fibonacci words were discovered. Knuth first presented it in 1968 [10]. However, according to Allouche and Shalit [11], pinpointing the actual source of the Fibonacci words is challenging. This is because the research on Fibonacci words was initially built on the basic ideas of Sturmian words, which are simple repeating sequences [12]. Fibonacci words, specifically, emerged as a structured variant of Sturmian words, whose lengths follow the Fibonacci sequence. By setting the initial terms $f(0) = x$ and $f(1) = y$, for all $n \geq 2$ by recursively applying $f(n) = f(n-1) \cdot f(n-2)$, the infinite sequence of strings is obtained. One of the combinatorial characteristics of Fibonacci words was studied by Aldo de Luca [13]. In 1986, Berstel compiled a list of all the substantiated findings involving Fibonacci words in a survey [14]. W. H. Chaun also studied about the Fibonacci words in detail [15], and analysed their symmetric nature [16], indexing nature which helps in generation [17] and the subwords of Fibonacci words [18]. In his discussion of factorization, G. Fici [19] addressed Fibonacci and related infinite terms. Kalpana Mahalingam et al. [20] recently studied Watson-Crick palindromes.

Lila Kari et al. [21] developed involutive Fibonacci words, driven by theoretical studies on DNA computing [22]. Additionally, they have studied the characteristics of indexed and bordered involutive Fibonacci words. By the end of the year 2022, they had examined primitivity in the context of involutive Fibonacci words [23]. Hannah and Sheena discovered involutive Fibonacci arrays and studied their properties [24].

1.1. Objective of this paper

In this paper one may find

- how the 0L-systems that generate involutive Fibonacci words are defined,
- how the fuzzy 0L-systems that generate fuzzy involutive Fibonacci words are defined,
- study some of the properties of the languages generated by 0L-systems and fuzzy 0L-systems defined in the previous sections and,
- analyze how bi-chromatic pictures can be generated by means of involutive Fibonacci words and prove that such languages of pictures are not recognizable.

1.2. Structure of this paper

This paper is organized in the following way: in Section 2, the essential terminology necessary for the study is reviewed. Sections 3 and 4 explain the L-systems that create the languages of involutive Fibonacci languages and fuzzy involutive Fibonacci languages, respectively. Section 5 looks at the features of involutive Fibonacci languages created by their specific 0L-systems, while Section 6 explores the closure properties of fuzzy involutive Fibonacci languages produced by their own 0L-systems. Finally, in Section 7, the generation of pictures using the involutive Fibonacci words is studied, and the languages of pictures generated by the involutive Fibonacci words are not recognizable is proved.

2. Preliminaries

In this section, various definitions that are necessary for our analysis are reviewed and discussed. For the fundamental definitions, one can refer to [21], [25], and [26].

Definition 1 [25]. An *alphabet*, represented by $\mathcal{S}$, is a non-empty and finite assortment of symbols. $|\mathcal{S}|$ denotes the count of symbols in $\mathcal{S}$.

Definition 2 [25]. A *string* or *word* is defined as a series of symbols derived from an alphabet. If $\mathcal{S}$ is an alphabet, then $\mathcal{S}^*$ denotes the assortment of strings generated by concatenating zero or more symbols from $\mathcal{S}$. A subset of $\mathcal{S}^*$ can be referred to as a *one-dimensional language*.

Definition 3 [25]. *Concatenation* of two strings s and t from $\mathcal{S}^*$, represented as $s \cdot t$ or st, is obtained by adding the symbols of t to the right end of s.(i.e.,) if $s = s_1 s_2 \cdots s_m$ and $t = t_1 t_2 \cdots t_n$, where $s_1, s_2, \cdots s_m, t_1, t_2, \cdots t_n \in \mathcal{S}$, then $st = s \cdot t = s_1 s_2 \cdots s_m t_1 t_2 \cdots t_n$.

Definition 4 [25]. The *length* of a word s refers to the total count of symbols it comprises, represented by $|s|$. If $|s| = 0$, then s is an *empty string*, commonly represented by ϵ. The set $\mathcal{S}^* \setminus \epsilon = \mathcal{S}^+$, where $\mathcal{S}^+$ denotes the assortment of all non-empty strings generated from the alphabet $\mathcal{S}$.

Definition 5 [25]. If $s \in \mathcal{S}^*$, then s^n is given by $s \cdot s \cdot s \cdots s$ (n-times), for all natural n.

Definition 6 [26]. A *0L-system* is expressed by a 3-tuple

$$G = \langle \mathcal{S}, P, \omega \rangle, \tag{1}$$

where:

- $\mathcal{S}$ - non-empty and finite collection, called the alphabet of the system;
- P is a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$. It is also known as the system's production set because it meets the following condition: $\forall a \in \mathcal{S}$, $\exists\, \rho \in \mathcal{S}^*$ such that $(a, \rho) \in P$;
- $\omega \in \mathcal{S}^*$ - axiom of the system.

To make a new string, the 0L-system applies rules of production to each symbol in the string at the same time, starting with the premise or axiom. This process is generally called as parallel rewriting. The 0L-system G defined above generates languages that are defined by

$$L(G) = \left\{ s \in \mathcal{S}^* : \omega \xrightarrow{*} s \right\} \tag{2}$$

Definition 7 [26]. A *fuzzy 0L-system* is defined as a 5-tuple

$$FG = \langle \mathcal{S}, P, \omega, I, \mu \rangle, \tag{3}$$

where:

- $\mathcal{S}$ is a non-empty and finite collection, called the *alphabet* of the system;
- P is a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$. It is also known as the system's *production set* because it satisfies the following condition: $\forall a \in \mathcal{S}$, $\exists\, \rho \in \mathcal{S}^*$ such that the pair $(a, \rho) \in P$;
- $\omega \in \mathcal{S}$ is the *axiom* of the system;
- $I = \{1, 2, \cdots, |P|\}$ is the set of labels assigned to the rules of production from P;

- $\mu : I \rightarrow [0, 1]$ assigns membership degree to the rules of production from P, i.e., $\forall\, l \in I$, $\exists\, m \in [0, 1] \subset \mathbb{R}$ such that $\mu(l) = m$.

The fuzzy language produced by the fuzzy 0L-system (3), given by $L(FG)$, is described as a fuzzy set in $\mathcal{S}^*$. A string $s \in \mathcal{S}^*$ is considered to be a member of this set in the following manner:

- Let d be a derivation that changes ω into s by means of a sequence of steps. Let $I' \subseteq I$ consist of labels that precisely correspond to the productions used in d. The strength of derivation d is defined by the smallest value of $\mu(l)$ for l in I', denoted by $s(d)$.

- Let $d_1, d_2, \cdots, d_r$ be distinct derivations of s. The membership degree of s in $L(FG)$ is as given here:

  – If $s = \omega$, then $\mu_{L(FG)}(s)$ gets a value of 1.

  – If there does not exist a derivation $\omega \xrightarrow{*} s$, then $\mu_{L(FG)}(s)$ takes the value zero.

  – $\mu_{L(FG)}(s)$ is equal to the maximum value of the strength of d_i, $s(d_i)$ for $i = 1, 2, \cdots, r$.

Definition 8 [21]. The *Fibonacci sequence* is a sequence of natural numbers in which each term is obtained by summing the two preceding terms. Specifically, $F(n) = F(n-1)+F(n-2)$ for all $n \geq 2$, with $F(0) = 0$ and $F(1) = 1$ established. Any element of this sequence is referred to as *a Fibonacci number*.

Definition 9 [21]. An *involution* is a function that functions as its own inverse. If τ^2 is equal to the identity function, where $\tau : X \rightarrow Y$ is a function, then it is an involution for any two non-empty sets X and Y. i.e., for any x in X, $\tau(\tau(x)) = x$.

Definition 10 [21]. A function $\tau : \mathcal{S}^* \rightarrow \mathcal{S}^*$ is termed as *morphism* over $\mathcal{S}^*$ if for $u = r_1r_2 \cdots r_k \in \mathcal{S}^*$, $\tau(u) = \tau(r_1r_2 \cdots r_k) = \tau(r_1)\tau(r_2) \cdots \tau(r_k), \forall\, r_i \in \mathcal{S}, 1 \leq i \leq k$.

Definition 11 [21]. A function $\tau : \mathcal{S}^* \rightarrow \mathcal{S}^*$ is defined as a *morphic involution* on $\mathcal{S}^*$ if it serves as an involution on $\mathcal{S}$ and is extended to function as a morphism on $\mathcal{S}^*$.

Definition 12 [21]. The *sequences of atom standard alternating τ-Fibonacci and reverse alternating τ-Fibonacci words*, denoted as $g(n)$ and $g'(n)$ over $\mathcal{S}^*$, are defined as $\{g(n)\}_{n\geq 0}$ and $\{g'(n)\}_{n\geq 0}$. The definitions are recursive: $g(0) = g'(0) = x$, $g(1) = g'(1) = y$, $g(n) = \tau(g(n-1)) \cdot g(n-2)$, and $g'(n) = g'(n-2) \cdot \tau(g(n-1))$ for $n \geq 2$ and for $x, y \in S$, where τ denotes a morphic involution over $\mathcal{S}^*$.

Definition 13 [21]. The *sequence of atom standard palindromic τ-Fibonacci and reverse palindromic τ-Fibonacci words*, denoted as $w(n)$ and $w'(n)$ over $\mathcal{S}^*$ are defined as $\{w(n)\}_{n\geq 0}$ and $\{w'(n)\}_{n\geq 0}$. The definitions are recursive: $w(0) = w'(0) = x$, $w(1) = w'(1) = y$, $w(n) = \tau(w(n-1)) \cdot \tau(w(n-2))$ and $w'(n) = \tau(w'(n-2)) \cdot \tau(w'(n-1))$ for $n \geq 2$ and for $x, y \in S$, where τ denotes a morphic involution over $\mathcal{S}^*$.

Definition 14 [21]. The *sequence of atom standard hairpin τ-Fibonacci and reverse hairpin τ-Fibonacci words*, denoted as $z(n)$ and $z'(n)$ over $\mathcal{S}^*$ are defined as $\{z(n)\}_{n\geq 0}$ and $\{z'(n)\}_{n\geq 0}$. The definitions are recursive: $z(0) = z'(0) = x$, $z(1) = z'(1) = y$ and $z(n) = z(n-1) \cdot \tau(z(n-2))$ and $z'(n) = \tau(z'(n-2)) \cdot z'(n-1)$ for $n \geq 2$ and for $x, y \in S$, where τ represents a morphic involution over $\mathcal{S}^*$.

3. L-systems to Construct Involutive Fibonacci Words

In this section, the definitions of the L-systems to construct the languages of Fibonacci and involutive Fibonacci words and arrays are given.

Definition 15. The sequence of standard (reverse) Fibonacci words can be derived using the 0L-system as follows: the 0L-system is defined as the 3-tuple

$$FG = \langle \mathcal{S}, P_F, \omega \rangle, \tag{4}$$

where:

- $\mathcal{S} = \{x, y\}$- alphabet of the system FG,
- P_F - a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$, known to be the set of productions of FFG given by:

$$P_F = \begin{cases} \{(x,y),(y,yx)\}, & \text{for standard Fibonacci words,} \\ \{(x,y),(y,xy)\}, & \text{for reverse Fibonacci words,} \end{cases}$$

- $\omega = x$ is the axiom of FFG.

Thus, the rules of production in P are given as follows: $x \to y$ and $y \to yx$ for standard Fibonacci words and $x \to y$ and $y \to xy$ for reverse Fibonacci words.

In a similar notion, the 0L-system to generate the languages of involutive Fibonacci words are defined below.

Definition 16. The sequence of involutive Fibonacci words can be derived using the 0L-system as follows: the 0L-system is defined as the 4-tuple

$$IFG = \langle \mathcal{S}, \tau, P_{IF}, \omega \rangle, \tag{5}$$

where:

- $\mathcal{S} = \{x, y\}$- alphabet of the system IFG,
- τ is a morphic involution over $\mathcal{S}^*$,
- P_{IF} - a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$, known to be the set of productions of IFG given by:

$$P_{IF} = \begin{cases} \{(x,y),(y,\tau(y)\,x),(\tau(x),\tau(y)),(\tau(y),y\,\tau(x))\}, & \text{for standard alternating } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,x\,\tau(y)),(\tau(x),\tau(y)),(\tau(y),\tau(x)\,y)\}, & \text{for reverse alternating } \tau\text{- Fibonacci words} \\ \{(x,y),(y,\tau(y)\,\tau(x)),(\tau(x),\tau(y)),(\tau(y),y\,x)\}, & \text{for standard palindromic } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,\tau(x)\,\tau(y)),(\tau(x),\tau(y)),(\tau(y),x\,y)\}, & \text{for reverse palindromic } \tau\text{- Fibonacci words} \\ \{(x,y),(y,y\,\tau(x)),(\tau(x),\tau(y)),(\tau(y),\tau(y)\,x)\}, & \text{for standard hairpin } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,\tau(x)\,y),(\tau(x),\tau(y)),(\tau(y),x\,\tau(y))\}, & \text{for reverse hairpin } \tau\text{- Fibonacci words,} \end{cases}$$

- $\omega = x$ is the axiom of IFG.

Definition 17. The above 0L-system IFG generates a language which is defined as

$$L(IFG) = \left\{ s \in \mathcal{S}^* : \omega \xrightarrow{*} s \right\} \tag{6}$$

Example 1. Consider the 0L-system $AFG = \langle \mathcal{S}, \tau, P_{AF}, \omega \rangle$, where $\mathcal{S} = \{x, y\}$, τ is defined by $\tau(x) = y$ and P_{AF} is the set of rules of production of the standard alternating τ- Fibonacci words, and $\omega = x$ is the initial symbol. Then the alternating τ- Fibonacci word $yyyyyyyy$ is derived using the rules as in (7): starting from x,

$$x \Rightarrow y \Rightarrow xx \Rightarrow yyy \Rightarrow xxxxx \Rightarrow yyyyyyyy \tag{7}$$

Thus, the language generated by the above AFG is

$$L(AFG) = \{x, y, xx, yyy, xxxxx, yyyyyyyy, \cdots\}. \tag{8}$$

4. L-systems to Construct Fuzzy Involutive Fibonacci Words

In this section, the definition of the fuzzy Fibonacci 0L-system is recalled from [27], followed by bringing in the concept of fuzzy involutive Fibonacci words.

Definition 18. [27] The *fuzzy standard Fibonacci 0L-system* is defined as the 5-tuple

$$FFG = \langle \mathcal{S}, P, \omega, I, \mu \rangle, \tag{9}$$

where:

- $\mathcal{S} = \{x, y\}$ is the alphabet of the fuzzy Fibonacci grammar FFG;
- P is a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$, known to be the set of productions of FFG given by:
$$P = \{(x, y), (y, yx)\}$$
- $\omega = x$ is the axiom of FFG;
- $I = \{1, 2\}$ is the set of labels for the rules of production in P;
- $\mu : I \to [0, 1]$ assigns membership degree to the rules of production from P, i.e., there exist two real numbers g_1 and $g_2 \in [0, 1]$ such that $\mu(1) = g_1$ and $\mu(2) = g_2$.

Thus, the rules of production in P, along with their membership degree, are given as follows:

$$1.\ x \to y,\ \mu(1) = g_1$$

$$2.\ y \to yx,\ \mu(2) = g_2$$

where $g_1, g_2 \in [0, 1]$.

Similarly, the fuzzy reverse Fibonacci 0L-system can be defined.

Definition 19. The *Fuzzy reverse Fibonacci 0L-system* is defined as the 5-tuple

$$FFG = \langle \mathcal{S}, P, \omega, I, \mu \rangle, \tag{10}$$

where:

- $\mathcal{S} = \{x, y\}$ is the alphabet of the fuzzy Fibonacci grammar FFG;

- P is a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$, known to be the set of productions of FFG given by:
$$P = \{(x, y), (y, xy)\}$$
- $\omega = x$ is the axiom of FFG;
- $I = \{1, 2\}$ is the set of labels for the rules of production in P;
- $\mu : I \to [0, 1]$ assigns membership degree to the rules of production from P, i.e., there exist two real numbers g_1 and $g_2 \in [0, 1]$ such that $\mu(1) = g_1$ and $\mu(2) = g_2$.

The definition of fuzzy involutive Fibonacci 0L-systems is given as follows.

Definition 20. The 0L-system that derives the sequence of fuzzy involutive Fibonacci words is defined as the 6-tuple
$$FIFG = \langle \mathcal{S}, \tau, P_{FIF}, \omega, I, \mu \rangle, \tag{11}$$
where:

- $\mathcal{S} = \{x, y\}$- alphabet of the system $FIFG$,
- τ is a morphic involution over $\mathcal{S}^*$,
- P_{FIF} - a non-empty and finite subset of $\mathcal{S} \times \mathcal{S}^*$, known to be the set of productions of $FIFG$ given by:

$$P_{FIF} = \begin{cases} \{(x,y),(y,\tau(y)\,x),(\tau(x),\tau(y)),(\tau(y),y\,\tau(x))\}, & \text{for standard alternating } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,x\,\tau(y)),(\tau(x),\tau(y)),(\tau(y),\tau(x)\,y)\}, & \text{for reverse alternating } \tau\text{- Fibonacci words} \\ \{(x,y),(y,\tau(y)\,\tau(x)),(\tau(x),\tau(y)),(\tau(y),y\,x)\}, & \text{for standard palindromic } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,\tau(x)\,\tau(y)),(\tau(x),\tau(y)),(\tau(y),x\,y)\}, & \text{for reverse palindromic } \tau\text{- Fibonacci words} \\ \{(x,y),(y,y\,\tau(x)),(\tau(x),\tau(y)),(\tau(y),\tau(y)\,x)\}, & \text{for standard hairpin } \tau\text{- Fibonacci words ,} \\ \{(x,y),(y,\tau(x)\,y),(\tau(x),\tau(y)),(\tau(y),x\,\tau(y))\}, & \text{for reverse hairpin } \tau\text{- Fibonacci words,} \end{cases}$$

- $\omega = x$ is the axiom of $FIFG$,
- $I = \{1, 2, 3, 4\}$ is the set of labels that represent the rules of production in P_{FIF}, and
- $\mu : I \to [0, 1]$ assigns membership degree to the rules of production from P_{FIF}, i.e., there exist four real numbers g_1, g_2, g_3 and $g_4 \in [0, 1]$ such that $\mu(1) = g_1, \mu(2) = g_2, \mu(3) = g_3$ and $\mu(4) = g_4$.

Definition 21. The 0L-system $FIFG$ defined above generates a language defined as
$$L(FIFG) = \left\{ s \in \mathcal{S}^* : \omega \xrightarrow{*} s \right\} \tag{12}$$
and s takes the membership value μ according to Definition 2.

Example 2. Let $r - FHFG = \langle \mathcal{S}, \tau, P_{r-HF}, \omega, I, \mu \rangle$ be a reverse fuzzy hairpin 0L-system, where $\mathcal{S} = \{x, y\}$, τ is a morphic involution on $\mathcal{S}^*$ defined by $\tau(x) = y$ and P_{r-HF} is the set of rules of production of the reverse hairpin τ- Fibonacci words given by:

$$\begin{array}{llllll} 1. & x \to y, & \mu(1) = 0.8, & 2. & y \to \tau(x)y, & \mu(2) = 0.6, \\ 3. & \tau(x) \to \tau(y), & \mu(3) = 0.4, & 4. & \tau(y) \to x\tau(y), & \mu(4) = 0.75, \end{array}$$

and $\omega = x$ is the initial symbol. Then the fuzzy reverse hairpin τ- Fibonacci word $(xxxyy, 0.4)$ is derived using the rules as in (13): starting from x,

$$x \xrightarrow{0.8} y \xrightarrow{0.6} \tau(x)y \xrightarrow{0.4} \tau(y)\tau(x)y \xrightarrow{0.4} x\tau(y)\tau(y)\tau(x)y \tag{13}$$

which is $xxxyy$.

Thus, the language generated by the above $r - FHFG$ is

$$L(r - FHFG) = \{(x, 1), (y, 0.8), (yy, 0.6), (xyy, 0.4), (xxxyy, 0.4), \cdots\} \tag{14}$$

5. Properties of the Languages Generated by the Involutive Fibonacci 0L-systems

In this section, some of the properties of involutive Fibonacci words are discussed. Throughout this section, let FIB denote the class of Fibonacci languages, $ALT\ FIB$ denote the class of alternating Fibonacci languages, $PAL\ FIB$ denote the class of palindromic Fibonacci languages, and $HP\ FIB$ denote the class of hairpin Fibonacci languages.

Theorem 1.

$$FIB \subsetneq ALT\ FIB, FIB \subsetneq PAL\ FIB \text{ and } FIB \subsetneq HP\ FIB$$

Proof: Let $FG = \langle \mathcal{S}, P, \omega \rangle$ be the Fibonacci 0L-system that generates $L(FG) \in FIB$. Let $IFG = \langle \mathcal{S}', \tau, P', \omega' \rangle$ be the involutive (alternating/ palindromic/ hairpin) Fibonacci 0L-system, where $\mathcal{S}' = \mathcal{S},\ P' = P,\ \omega' = \omega$ and τ is an identity map in $\mathcal{S}'$. Then, $L(IFG) = L(FG)$ and therefore, $FIB \subset ALT\ FIB, FIB \subset PAL\ FIB$ and $FIB \subset HP\ FIB$.

Consider $L(AFG^*) = \{x, y, xx, yyy, xxxxx, \cdots\} \in ALT\ FIB$, generated by the fuzzy alternating 0L-system $AFG^* = \langle \mathcal{S}, \tau, P, \omega \rangle$, with $\mathcal{S} = \{x, y\}, \tau$ is a morphic involution in $\mathcal{S}$ given by $\tau(x) = y$, the set of alternating Fibonacci rules of production
$P = \{x \to y, y \to \tau(y)x, \tau(x) \to \tau(y), \tau(y) \to y\tau(x)\}$, and $\omega = x$. This 0L-system does not generate any Fibonacci language, and hence $FIB \subsetneq ALT\ FIB$.

Similarly, the other two instances can be shown. ■

Theorem 2. $ALT\ FIB, PAL\ FIB$ and $HP\ FIB$ are not closed under union, intersection, concatenation, and complementarity.

Theorem 3. $ALT\ FIB, PAL\ FIB$ and $HP\ FIB$ are closed under homomorphism.

Proof: Let IFL be an involutive Fibonacci language (alternating, palindromic, or hairpin) generated by the 0L-system $IFG = \langle \mathcal{S} = \{x, y\}, \tau, P_{IF}, \omega = x \rangle$. Let $h : \mathcal{S} \to \mathcal{S}'$ be a homomorphism. In the 0L-system IFG, the following changes are carried out to get the new system IFG':

- Replace the symbols a by $h(a)$ respectively, where $a \in \mathcal{S}$ and $h(a) \in \mathcal{S}'$ in P_{IF};
- Replace the morphic involution τ by τ', where $\tau(a) = \tau'(h(a)), \forall a \in \mathcal{S}$;
- Replace $\omega = x$ by $\omega' = h(x)$

Thus the new 0L-system $IFG' = \langle \mathcal{S}', \tau', P'_{IF}, \omega' \rangle$ generates an involutive Fibonacci language L' over $\mathcal{S}'$. ■

Theorem 4. $ALT\ FIB, PAL\ FIB$ and $HP\ FIB$ are closed under inverse homomorphism.

6. Closure Properties of the Languages Generated by the Fuzzy Involutive Fibonacci 0L-systems

In this section, some of the closure properties of fuzzy involutive Fibonacci words generated by their respective fuzzy 0L-systems are discussed. Throughout this section, $FIFL$ denotes the language of fuzzy involutive (alternating, palindromic, or hairpin) Fibonacci words generated by their respective 0L-system.

Theorem 5. The family of $FIFL$ is not closed under union, intersection, concatenation, and complementarity.

The lack of closure is proved as follows by providing specific counterexamples. Consider the alternating Fibonacci languages $1-FAFL_1 = \{(x,1),(y,0.5),(xx,0.4),(yyy,0.3),\cdots\}$ and $1-FAFL_2 = \{(x,1),(y,0.4),(yx,0.3),(yxy,0.4),\cdots\}$. Then, $1-FAFL_1 \cup 1-FAFL_2 = \{(x,1),(y,0.5),(xx,0.4),(yx,0.3),(xy,0),(yy,0),(yxy,0.4),(yyy,0.3),\cdots\}$, $1-FAFL_1 \cap 1-FAFL_2 = \{(x,1),(y,0.4),(xx,0),(yx,0),(yxy,0),(yyy,0),\cdots\}$, $1-FAFL_1 \cdot 1-FAFL_2 = \{(xx,1),(xy,0.4),(xyx,0.3),(yx,0.5),(yy,0.4),(yyx,0.3),\cdots\}$ and $1-FAFL_1^C = \{(x,0),(y,0.5),(xx,0.6),(xy,1),(yx,1),(yy,1),(yyy,0.7),(xxx,1),\cdots\}$ which are not alternating Fibonacci languages. Similarly, the other involutive Fibonacci languages can be checked.

Theorem 6. The family of $FIFL$ is closed under homomorphism.

Proof: Let $1-IFL$ be the fuzzy involutive Fibonacci language generated by the fuzzy involutive Fibonacci 0L-system $1-FIFG = \langle \mathcal{S} = \{x,y\}, \tau, P_{IF}, \omega = x, l, \mu \rangle$. Let $h : \mathcal{S} \to \mathcal{S}'$ be a homomorphism. In the 0L-system $1-FIFG$, the following changes are carried out to get $1-FIFG'$:

- Replace the symbols a with $h(a)$, $\forall a \in \mathcal{S}$ and $h(a) \in \mathcal{S}'$ in P_{IF};
- Replace the morphic involution τ by τ' where $\tau(a) = \tau'(h(a))$, for all $a \in \mathcal{S}$;
- Define the new membership function μ' by setting $\mu'(1) = \mu(1), \mu'(2) = \mu(2), \mu'(3) = \mu(3)$ and $\mu'(4) = \mu(4)$.

Thus, the new fuzzy 0L-system $1-FIFG' = \langle \mathcal{S}', \tau', P'_{IF}, \omega', l, \mu' \rangle$ generates a fuzzy involutive Fibonacci language L' over $\mathcal{S}'$. ■

Theorem 7. The family of $FIFL$ is closed under inverse homomorphism.

7. Pictures Generated Using Involutive Fibonacci Words

In this section the properties of the pictures (arrays) generated using the Fibonacci and involutive Fibonacci words are studied.

7.1. Generation of pictures

The pictures are generated in terms of the following steps:

- **Step 1:** First generate the involutive (alternating, palindromic, or hairpin) Fibonacci words of order n, say, $\omega(n)$ which is of length $F(n)$, where $n \in \mathbb{N} \setminus \{1\}$
- **Step 2:** Fix the value of $m \in \mathbb{N}$, the scale of the square array $a_{m,m}$ that is to be generated.

- **Step 3:** Consider the word $\omega(n) = \omega_1\omega_2\cdots\omega_n,\ \omega_i \in \{x,y\}$. Each ω_i can be labeled with the labels $0, 1, \cdots, m-1$.
- **Step 4:** Fill the square array $a_{i,j}$ with the symbols whose index is $(i+j) \mod F(n)$.
- **Step 5:** The resultant array is the required picture generated by the involutive Fibonacci word $\omega(n)$, where ω denotes either alternating, palindromic, or hairpin Fibonacci words.

The pictures generated in the above-said manner from the involutive Fibonacci words of order 7 and scale are fixed as 50, keeping black as x and white as y. These pictures are given in Table 1 in [29].

The assortment of pictures generated by an involutive (alternating, palindromic, or hairpin) Fibonacci word is called the *language* generated from the involutive (alternating, palindromic, or hairpin) Fibonacci words.

Example 3. The language generated by the reverse palindromic Fibonacci word xxy, whose morphic involution τ is defined as $\tau(x) = y$, of order 3 is

$$\left\{ x, \begin{matrix} x & x \\ x & x \end{matrix}, \begin{matrix} x & x & y \\ x & y & x \\ y & x & x \end{matrix}, \begin{matrix} x & x & y & x \\ x & y & x & x \\ y & x & x & y \\ x & x & y & x \end{matrix}, \begin{matrix} x & x & y & x & x \\ x & y & x & x & y \\ y & x & x & y & x \\ x & x & y & x & x \\ x & y & x & x & y \end{matrix}, \begin{matrix} x & x & y & x & x & y \\ x & y & x & x & y & x \\ y & x & x & y & x & x \\ x & x & y & x & x & y \\ x & y & x & x & y & x \\ y & x & x & y & x & x \end{matrix}, \cdots \right\} \tag{15}$$

Theorem 8. The languages of pictures generated from the involutive Fibonacci words are not recognizable.

Proof: This is evident from the horizontal iteration lemma, also called the pumping lemma for two-dimensional languages, which is considered the necessary condition for recognizability [28]. It states that for a recognizable language L, there exists a function $f : \mathbb{N} \to \mathbb{N}$, such that if $p \in L$ and the number of columns in p exceeds f(number of rows in p), then there exist some pictures a, b and c with $|a \ominus b|_{col} \leq f(|p|_{row})$ and $|c|_{col} > 1$ so that $p = a \ominus b^{i\,\ominus} \ominus c \in L, i \geq 0$.

Since b, the subpicture of p needs to be pumped up in order to belong to L, making it recognizable, which seems to fail in the scenario of the languages generated by the involutive Fibonacci words.

This contradiction proves that L does not satisfy the pumping lemma and hence L is not recognizable. An illustration of this result is given in [29]. ■

8. Conclusions

In this paper, the 0L-systems generating the languages of standard and reverse alternating, palindromic, and hairpin Fibonacci words are defined, and their properties are discussed. Also, the 0L-systems for fuzzy variants of these languages are introduced, and their characteristics are examined.

As an application, how involutive Fibonacci words can be used to generate picture languages is demonstrated. It is proven that these languages are not recognizable. Future research will focus on studying involutive Fibonacci arrays generated by 0L-systems.

References

[1] A. THUE, *Über unendliche Zeichenreihen*, Norske Videnskabers Selskabs Skrifter, Mathematisk-Naturvidenskabelig Klasse **7**, 1906, pp. 1–22.

[2] A. THUE and T. NAGELL, *Selected Mathematical Papers of Axel Thue*, Universitetsforlaget, Oslo, 1977.

[3] M. LOTHAIRE, *Combinatorics on Words*, Addison-Wesley, Reading, MA, USA, 1983.

[4] L. KARI and K. MAHALINGAM, *Involutively bordered words*, International Journal of Foundations of Computer Science **18**(5), 2022, pp. 1089–1106.

[5] E. CZEIZLER, L. KARI and S. SEKI, *On a special class of primitive words*, Theoretical Computer Science **411**, 2010, pp. 617–630.

[6] S. S. YU, *Languages and Codes*, Lecture Notes 402, Department of Computer Science, National Chung-Hsing University, Taichung, Taiwan, 2005.

[7] D. E. KNUTH, J. H. MORRIS (JR.) and V. R. PRATT, *Fast pattern matching in strings*, SIAM Journal on Computing, **6**(2), 1977, pp. 323–350.

[8] R. COLE, M. CROCHEMORE, Z. GALIL, L. GASIENIEC, R. HARIHARAN, S. MUTHUKRISHNAN, K. PARK and W. RYTTER, *Optimally fast parallel algorithms for preprocessing and pattern matching in one and two dimensions*, Proceedings of IEEE 34$^{\text{th}}$ Annual Foundations of Computer Science Conference, Palo Alto, CA, USA, 1993, pp. 248–258.

[9] S. H. GEIZHALS and D. SOKOL, *Finding maximal 2-dimensional palindromes*, Information and Computation **266**, 2019, pp. 161–172.

[10] D. E. KNUTH, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*, 1$^{\text{st}}$ ed., Addison-Wesley, Reading, MA, 1968.

[11] J. P. ALLOUCHE and J. SHALLIT, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, MA, 2003.

[12] M. MORSE and G. A. HEDLUND, *Symbolic dynamics*, American Journal of Mathematics **62**(1), 1940, pp. 1–42.

[13] A. DE LUCA, *A combinatorial property of the Fibonacci words*, Information Processing Letters **12**(4), 1981, pp. 193–195.

[14] J. BERSTEL, *Fibonacci words - A survey*, in Combinatorics on Words, M. Lothaire, Ed., Springer, Berlin, Heidelberg, pp. 13–27, 1986.

[15] W. F. CHUAN, *Fibonacci words*, The Fibonacci Quarterly **30**(1), 1992, pp. 68–76.

[16] W. F. CHUAN, *Symmetric Fibonacci words*, The Fibonacci Quarterly **31**(3), 1993, pp. 251–255.

[17] W. F. CHUAN, *Generating Fibonacci words*, The Fibonacci Quarterly **33**(2), 1995, pp. 104–112.

[18] W. F. CHUAN, *Subwords of the golden sequence and the Fibonacci words*, in Applications of Fibonacci Numbers, G. E. Bergum, A. N. Philippou and A. F. Horadam, Eds., Springer, Dordrecht, pp. 73–84, 1996.

[19] G. FICI, *Factorizations of the Fibonacci infinite word*, Journal of Integer Sequences **18**(9), 2015, pp. 1–4.

[20] K. MAHALINGAM, A. MAITY and P. PANDOH, *Counting (Watson-Crick) palindromes in Watson-Crick conjugates*, Information and Computation **285**, 2022, paper 104863.

[21] L. KARI, M. S. KULKARNI, K. MAHALINGAM and Z. WANG, *Involutive Fibonacci words*, Journal of Automata, Languages and Combinatorics **26**, 2021, pp. 255–280.

[22] L. KARI, R. KITTO and G. THIERRIN, *Codes, involutions and DNA encodings*, in Formal and Natural Computing: Essays Dedicated to Grzegorz Rozenberg, W. Brauer, H. Ehrig, J. Karhumäki and A. Salomaa, Eds., Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, pp. 376–393, 2002.

[23] L. KARI, K. MAHALINGAM, P. PANDOH and Z. WANG, *Primitivity of Atom Watson-Crick Fibonacci words*, Journal of Automata, Languages and Combinatorics **27**, 2021, pp. 255–280.

[24] H. BLASIYUS and D. K. S. CHRISTY, *Palindromes in involutive Fibonacci arrays*, Journal of Supercomputing **80**(15), 2024, pp. 22520–22547.

[25] P. LINZ and S. H. RODGER, *An Introduction to Formal Languages and Automata*, 7^{th} ed., Jones & Bartlett, Sudbury, MA, 2023.

[26] A. LINDENMAYER, *Mathematical models for cellular interactions in development I. Filaments with one-sided inputs*, Journal of Theoretical Biology **18**, 1968, pp. 280–299.

[27] H. BLASIYUS, D. K. S. CHRISTY and D. G. THOMAS, *Fuzzy Fibonacci arrays and its application — symmetrical Fibonacci dielectric multilayers in transmission of light*, The Fibonacci Quarterly **63**(4), 2025, pp. 717–735.

[28] D. GIAMMARRESI and A. RESTIVO, *Two-dimensional languages*, in Handbook of Formal Languages: Volume 3 Beyond Words, G. Rozenberg and A. Salomaa, Eds., Springer, Berlin, Heidelberg, pp. 215–267, 1997.

[29] H. BLASIYUS and D. K. S. CHRISTY, *Supplementary Material for Languages of Fuzzy Involutive Fibonacci words*, Available:
`https://drive.google.com/file/d/1ehJua70ropx_NeJhCIW5Ip6kbElBBvwm/view?usp=sharing`.